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EMBEDDING SPANNING BOUNDED DEGREE GRAPHS IN RANDOMLY PERTURBED GRAPHS

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Abstract. We study the model $G_\alpha \cup G(n,p)$ of randomly perturbed dense graphs, where G_α is any n-vertex graph with minimum degree at least αn and $G(n,p)$ is the binomial random graph. We introduce a general approach for studying the appearance of spanning subgraphs in this model using absorption. This approach yields simpler proofs of several known results. We also use it to derive the following two new results.

For every $\alpha > 0$ and $\Delta \geqslant 5$, and every n-vertex graph F with maximum degree at most Δ, we show that if $p = \omega(n^{-2/(\Delta+1)})$, then $G_\alpha \cup G(n,p)$ with high probability contains a copy of F. The bound used for p here is lower by a log-factor in comparison to the conjectured threshold for the general appearance of such subgraphs in $G(n,p)$ alone, a typical feature of previous results concerning randomly perturbed dense graphs.

We also give the first example of graphs where the appearance threshold in $G_\alpha \cup G(n,p)$ is lower than the appearance threshold in $G(n,p)$ by substantially more than a log-factor. We prove that, for every $k \geqslant 2$ and $\alpha > 0$, there is some $\eta > 0$ for which the kth power of a Hamilton cycle with high probability appears in $G_\alpha \cup G(n,p)$ when $p = \omega(n^{-1/k-\eta})$. The appearance threshold of the kth power of a Hamilton cycle in $G(n,p)$ alone is known to be $n^{-1/k}$, up to a log-term when $k = 2$, and exactly for $k > 2$.

§1. *Introduction and results.* Many important results in Extremal Graph Theory and in Random Graph Theory concern the appearance of spanning subgraphs in dense graphs and in random graphs, respectively. In Extremal Graph Theory, minimum degree conditions forcing the appearance of such subgraphs are studied. For example, Dirac's Theorem [13], one of the cornerstones of Extremal Graph Theory, states that an n-vertex graph with minimum degree at least $n/2$ has a Hamilton cycle when $n \geqslant 3$. In Random Graph Theory, on the other hand, bounds are sought on the probability threshold for the appearance of subgraphs in a random graph. Let $G(n,p)$ be the *binomial random graph* model with vertex set $[n]$, where each possible edge is chosen independently at random with probability p. We say that $G(n,p)$ has some property $\mathcal{P}$ *with high probability* (whp) if $\lim_{n\to\infty} \mathbb{P}[G(n,p) \in \mathcal{P}] = 1$. A key result by Pósa [35] and Koršunov [26] is that $G(n,p)$ with high probability contains a Hamilton cycle if $p = \omega(\log n/n)$, whereas if $p = o(\log n/n)$, then $G(n,p)$ with high probability does not. Here, we write $p(n) = \omega(f(n))$ to signify $p(n)/f(n) \to \infty$, and $p(n) = o(f(n))$ to signify $p(n)/f(n) \to 0$.

The study of randomly perturbed graphs combines these two approaches by taking the union of a graph satisfying some minimum degree condition and a random graph $G(n,p)$. The goal is then to determine which minimum degree conditions and edge probabilities suffice to guarantee some given subgraph with high probability. Bohman *et al.* [8], who pioneered

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the study of randomly perturbed graphs, proved that for every $\alpha > 0$ the union of every n-vertex graph with minimum degree at least αn and a random graph $G(n, p)$ with $p = \omega(1/n)$ contains whp a Hamilton cycle. This result shows that, compared to Dirac's Theorem, a much smaller minimum degree condition suffices in a randomly perturbed graph, and compared to the random graph $G(n, p)$ alone a log-term improvement in the edge probability is possible.

The recent increased interest in randomly perturbed graphs sparked a collection of results of a similar flavour, typically featuring a small linear minimum degree condition and a log-term improvement in the edge probability. In this paper, we contribute to this body of research by developing a new general method for establishing such results for spanning subgraphs. Our approach uses an absorbing method. We show that this new approach gives simpler proofs of a number of known results, whose original proofs often use the regularity method and are therefore technically more complex. It also allows us to give strong new results concerning powers of Hamilton cycles and general bounded degree spanning subgraphs in randomly perturbed graphs. In particular, our result on powers of Hamilton cycles provides the first example for graphs with an $n^{\Omega(1)}$ improvement in the edge probability compared to $G(n, p)$. A similar phenomenon was already discovered in the context of hypergraphs by McDowell and Mycroft [30], which we will return to in our concluding remarks.

Before discussing our techniques and results in more detail, we set our work in context by summarising related results in random graphs and randomly perturbed graphs.

1.1. *Thresholds in $G(n, p)$.* We say that the function $\hat{p} : \mathbb{N} \to [0, 1]$ is a *threshold* for a graph property $\mathcal{P}$, if

$$\lim_{n\to\infty} \mathbb{P}[G(n, p) \in \mathcal{P}] = \begin{cases} 0 & \text{whenever } p = o(\hat{p}), \text{ and} \\ 1 & \text{whenever } p = \omega(\hat{p})\,. \end{cases}$$

If only the latter is known to be true, then we say that $\hat{p}$ is an upper bound for the threshold for $\mathcal{P}$ in $G(n, p)$. Containing a graph as a (not necessarily induced) subgraph is a monotone property and therefore it has a threshold by a result of Bollobás and Thomason [10]. In the following, we will focus on spanning subgraphs.

In their seminal work, Erdős and Rényi [15] proved that the threshold for perfect matchings in $G(n, p)$ is $\log n/n$. Pósa [35] and Koršunov [26] independently showed that the property of having a Hamilton cycle has the same threshold.

The problem of finding powers of Hamilton cycles as a subgraph is generally considered a stepping stone towards results for more general spanning subgraphs. The kth *power* $G^{(k)}$ of a graph G is the graph obtained from G by connecting all vertices at distance at most k. Kühn and Osthus [29] observed that the threshold in $G(n, p)$ for the kth power of a Hamilton cycle when $k \geqslant 3$ is $n^{-1/k}$; this follows from a general embedding theorem due to Riordan [36] (see Theorem 2.4). Similarly, the threshold of the square of a Hamilton cycle is conjectured to be $n^{-1/2}$, but this is still open. Currently, the best known upper bound, by Nenadov and Škorić [33], is off by a $O(\log^4 n)$-factor from this conjectured threshold.

For a graph H, an *H-factor* on n vertices is the vertex disjoint union of copies of H with n vertices in total. An *almost H-factor* in an n-vertex graph G is a subgraph of G that is an H-factor on $(1 - \varepsilon)n$ vertices. A breakthrough result was achieved by Johansson *et al.* [23] who showed that the threshold for a $K_{\Delta+1}$-factor, that is $\frac{n}{\Delta+1}$ vertex-disjoint copies of $K_{\Delta+1}$, is given by

$$p_\Delta = \left(\frac{\log^{1/\Delta} n}{n}\right)^{2/(\Delta+1)}.$$

In fact, their result concerns, more generally, H-factors for strictly balanced graphs H. The 1-*density* of a graph H on at least two vertices is

$$m_1(H) = \max_{H' \subseteq H, v(H')>1} \frac{e(H')}{v(H') - 1},$$

and a graph is called *strictly balanced* if H is the only maximiser in $m_1(H)$. Johansson *et al.* [23] proved that for factors of strictly balanced graphs H, the threshold is $n^{-1/m_1(H)} \log^{1/e(H)} n$. Gerke and McDowell [20], on the other hand, showed that for certain (but not all) graphs H which are not strictly balanced, this threshold is $n^{-1/m_1(H)}$.

Let us now turn to larger classes of graphs. For bounded degree spanning trees, the second author [32] showed that, for each fixed Δ, $\log n/n$ is the appearance threshold for single spanning trees with maximum degree at most Δ (see also [31]).

More generally, let $\mathcal{F}(n, \Delta)$ be the family of graphs on n vertices with maximum degree at most Δ. For some constant C, Alon and Füredi [3] proved that, if $p \geqslant C(\log n/n)^{1/\Delta}$, then $G(n, p)$ contains any single graph from $\mathcal{F}(n, \Delta)$ whp. This is far from optimal and, since the clique-factor is widely believed to have the highest appearance threshold among the graphs in $\mathcal{F}(n, \Delta)$, the following well-known conjecture is natural.

CONJECTURE 1.1. If $\Delta \in \mathbb{N}$, $F \in \mathcal{F}(n, \Delta)$ and $p = \omega(p_\Delta)$, then $G(n, p)$ whp contains a copy of F.

For $\Delta = 2$, this conjecture was very recently resolved by Ferber *et al.* [16], who in fact showed a stronger *universality* statement, where all graphs in $\mathcal{F}(n, \Delta)$ are found simultaneously. For larger Δ, Riordan [36] gave a general result (see Theorem 2.4), which requires an edge probability within a factor of $n^{\Theta(1/\Delta^2)}$ from p_Δ. The current best result in the direction of Conjecture 1.1 is the following *almost* spanning version by Ferber *et al.* [17].

THEOREM 1.2 (Ferber *et al.* [17]). *Let $\varepsilon > 0$ and $\Delta \geqslant 5$. For every $F \in \mathcal{F}((1-\varepsilon)n, \Delta)$ and $p = \omega(p_\Delta)$ the random graph $G(n, p)$ whp contains a copy of F.*

The approach in [17] is based on ideas from Conlon *et al.* [12], who proved a stronger universality statement for the almost spanning case while using the edge probability $n^{-1/(\Delta-1)} \log^5 n$. Theorem 1.2 for $\Delta = 3$ was thus already known (up to a log-factor), whereas the case for $\Delta = 4$ remains open. For spanning subgraphs, very recently, Ferber and Nenadov [18] showed that for $p \geqslant (\log^3 n/n)^{1/(\Delta-1/2)}$ the random graph $G(n, p)$ whp contains all graphs in $\mathcal{F}(n, \Delta)$ universally.

In the almost spanning case, the log-term in p_Δ is expected to be redundant [17], but this remains open. In this paper, we will show that the log-term in p_Δ is redundant, even in the spanning case, if we add $G(n, p)$ to a deterministic graph with linear minimum degree.

1.2. *Randomly perturbed graphs.* Bohman *et al.* [9] introduced the following model of randomly perturbed graphs. For $\alpha \in (0, 1)$ and an integer n, we first let G_α be any n-vertex graph with minimum degree at least αn. We then reveal more edges among the vertices of G_α independently at random with probability p. The resulting graph $G_\alpha \cup G(n, p)$ is a *randomly perturbed graph* and we are interested in its properties. In particular, research has focused on comparing thresholds in $G_\alpha \cup G(n, p)$ to thresholds in $G(n, p)$.

Again, we concentrate on spanning subgraphs. Note that the existence of such subgraphs in $G_\alpha \cup G(n, p)$ is a monotone property (in $G(n, p)$), and thus has a threshold. Of course,

if $\alpha \geqslant 1/2$, then G_α is itself Hamiltonian by Dirac's Theorem. For $\alpha \in (0, 1/2)$, Bohman *et al.* [9] showed the existence of some $c = c(\alpha) > 0$ so that, if $p = c/n$, then, for any G_α, there is a Hamilton cycle in $G_\alpha \cup G(n, p)$ whp. They also proved that this is optimal: there exists some $c' > 0$ so that there are graphs G_α such that $G_\alpha \cup G(n, c'/n)$ is not Hamiltonian whp. Comparing this threshold to the threshold for Hamiltonicity in $G(n, p)$, we note an extra factor of $\log n$ in the latter. This $\log n$ term is necessary to guarantee minimum degree at least 2 in $G(n, p)$—otherwise clearly no Hamilton cycle exists. In the model $G_\alpha \cup G(n, p)$, however, this already holds in G_α alone.

Krivelevich *et al.* [28] studied the corresponding problem for the containment of spanning trees of maximum degree Δ in $G_\alpha \cup G(n, p)$. For $p = c(\varepsilon, \Delta)/n$, it is already possible to find any *almost* spanning bounded degree tree on $(1 - \varepsilon)n$ vertices in $G(n, p)$ [4]. The addition of G_α then ensures there are no isolated vertices, and Krivelevich *et al.* [28] showed that this indeed allows every vertex to be incorporated into the embedding. They thus prove that, for $\alpha > 0$, maximum degree Δ and $p = c(\alpha, \Delta)/n$, every spanning bounded degree tree is contained in $G_\alpha \cup G(n, p)$.

Very recently, Balogh *et al.* [5] determined the threshold of appearance for general factors in the model $G_\alpha \cup G(n, p)$. They proved that for every H, if $p = \omega(n^{-1/m_1(H)})$, then $G_\alpha \cup G(n, p)$ contains an H-factor whp. Comparing this to the result of Johansson *et al.* [23], we observe again a saving of a log-term. For the graphs H covered by the result of Gerke and McDowell [20], on the other hand, we see that the thresholds in $G_\alpha \cup G(n, p)$ and in $G(n, p)$ are the same.

Other monotone properties considered in the randomly perturbed graph model include containing a fixed sized clique, having small diameter, being k-connected [8] and being non-2-colourable [37].

1.3. *Our results.* Our main contribution to the study of randomly perturbed graphs is the introduction of a new approach for obtaining results concerning spanning subgraphs. The basic idea is to use some random edges with the assistance of the deterministic edges to create so-called *reservoir sets*. Our key technical result is Theorem 3.3, which gives a condition for applying this method to spanning subgraphs. We defer the statement of this result along with the necessary definitions to § 3.

Using our method, we analyse the model $G_\alpha \cup G(n, p)$ with respect to the containment of spanning bounded degree graphs, addressing a problem which was highlighted by Krivelevich *et al.* in the concluding remarks of [28]. We obtain the following result.

THEOREM 1.3. *Let $\alpha > 0$ be a constant, $\Delta \geqslant 5$ be an integer and G_α be a graph with minimum degree at least αn. Then, for every $F \in \mathcal{F}(n, \Delta)$ and $p = \omega(n^{-2/(\Delta+1)})$, whp $G_\alpha \cup G(n, p)$ contains a copy of F.*

Our bound on p in Theorem 1.3 is best possible in the following sense. In the case where F is a $K_{\Delta+1}$-factor on n vertices and G_α is a complete bipartite graph with parts of size αn and $(1 - \alpha)n$, we need to find an almost spanning $K_{\Delta+1}$-factor on $(1 - \alpha(\Delta + 1))n$ vertices in $G(n, p)$. This can easily be shown to require $p = \Omega(n^{-2/(\Delta+1)})$. Note in addition that the edge probability used in Theorem 1.3 is lower by a log-term in comparison to the anticipated threshold for the graph F to appear in $G(n, p)$ (see Conjecture 1.1).

Our second result deals with powers of Hamilton cycles. Here we can save a polynomial factor $n^{\Omega(1)}$ compared to the threshold $n^{-1/k}$ in $G(n, p)$.

THEOREM 1.4. *For each $k \geqslant 2$ and $\alpha > 0$, there is some $\eta > 0$, such that if G_α is an n-vertex graph with minimum degree at least αn, then $G_\alpha \cup G(n, n^{-1/k-\eta})$ whp contains the kth power of a Hamilton cycle.*

It was proved by Komlós *et al.* [25] that G_α on its own contains the kth power of a Hamilton cycle, provided that $\alpha \geqslant k/(k+1)$ and $v(G_\alpha)$ is large enough. Bedenknecht *et al.* [6] showed that for any $k \geqslant 3$ there is an η so that $G_\alpha \cup G(n, n^{-1/k-\eta})$ whp contains the kth power of a Hamilton cycle if $\alpha > c_k$ for some absolute constant $c_k > 0$.

Bennett *et al.* [7] provide the following lower bound. With G_α the complete bipartite graph with αn and $(1-\alpha)n$ vertices in the classes, they show that p has to be at least $n^{-1/k(1-2\alpha)}$ for $G_\alpha \cup G(n,p)$ to contain the kth power of a Hamilton cycle. It would be interesting to determine the optimal dependence between α, k and η in Theorem 1.4.

Organisation. We finish this section by providing some further notation, before outlining our general embedding method for randomly perturbed graphs in § 3. We then prove Theorem 1.4, the less technical of our implementations of this method, in § 4. Theorem 1.3 is proved in § 5, with the proofs of two auxiliary lemmas given in § 6. Finally, we make some concluding remarks and sketch how our methods can give simpler proofs of other results in the literature concerning randomly perturbed graphs in § 7.

Notation. A graph G has vertex set $V(G)$, edge set $E(G)$ and we let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a vertex $v \in V(G)$, $N_G(v)$ is the set of neighbours of v in G, and for a vertex set $A \subseteq V(G)$, $N_G(A) = (\cup_{v\in A} N_G(v)) \setminus A$. Where no confusion is likely to occur, we simply write $N(v)$ and $N(A)$. For graphs G and H, $G \cap H$ is the graph on vertex set $V(G) \cap V(H)$ with edge set $E(G) \cap E(H)$. For a graph G, and a vertex set $A \subseteq V(G)$, $G[A]$ is the induced subgraph of G on A, and $G - A = G[V(G) \setminus A]$.

§2. *Tools.* Our results concern the embedding of certain graphs F in $G_\alpha \cup G(n,p)$. For obtaining such an embedding, our first step will always be to embed an almost spanning subgraph F^* of F, and our second step then (working in an auxiliary graph on $[2n]$ vertices) extends this to an embedding of F.

For the second step, we shall use the following hypergraph matching theorem of Aharoni and Haxell [1]. The setup will be as follows. $F \setminus F^*$ consists of t well-separated subgraphs $S_1, \dots, S_t$ of F, and we shall encode all valid embeddings of S_i that extend the embedding of F^* as the edges of a hypergraph L_i. The goal then is to find a hypergraph matching using exactly one edge from each L_i. A hypergraph is *r-uniform* if each of its edges has cardinality r.

THEOREM 2.1 (Hall's condition for hypergraphs [1])). *Let $\{L_1, \dots, L_t\}$ be a family of s-uniform hypergraphs on the same vertex set. If, for every $\mathcal{I} \subseteq [t]$, the hypergraph $\bigcup_{i\in\mathcal{I}} L_i$ contains a matching of size greater than $s(|\mathcal{I}|-1)$, then there exists a function $g : [t] \to \bigcup_{i=1}^t E(L_i)$ such that $g(i) \in E(L_i)$ and $g(i) \cap g(j) = \emptyset$ for $i \neq j$.*

When we want to use this theorem, we need to verify the condition on L_i. For this purpose, we shall use Janson's inequality (see, e.g., [22, Theorem 2.18]).

LEMMA 2.2 (Janson's inequality). *Let $p \in (0,1)$ and consider a family $\{H_i\}_{i\in\mathcal{I}}$ of subgraphs of the complete graph on the vertex set $[n] = \{1, \dots, n\}$. For each $i \in \mathcal{I}$, let X_i denote the indicator random variable for the event that $H_i \subseteq G(n,p)$ and, write $H_i \sim H_j$ for*

each ordered pair $(i, j) \in \mathcal{I} \times \mathcal{I}$ *with* $i \neq j$ *if* $E(H_i) \cap E(H_j) \neq \emptyset$. *Then, for* $X = \sum_{i \in \mathcal{I}} X_i$, $\mathbb{E}[X] = \sum_{i \in \mathcal{I}} p^{e(H_i)}$,

$$\delta = \sum_{H_i \sim H_j} \mathbb{E}[X_i X_j] = \sum_{H_i \sim H_j} p^{e(H_i)+e(H_j)-e(H_i \cap H_j)}$$

and any $0 < \gamma < 1$, *we have*

$$\mathbb{P}[X \leqslant (1-\gamma)\mathbb{E}[X]] \leqslant \exp\left(-\frac{\gamma^2 \mathbb{E}[X]^2}{2(\mathbb{E}[X]+\delta)}\right).$$

This result will also be useful for the first step described above, in which we embed an almost spanning subgraph. In particular, the appearance of almost H-factors in $G(n, p)$ for $p \geqslant Cn^{-1/m_1(H)}$ is a straightforward consequence of Janson's inequality (see, e.g., [22, Theorem 4.9]). Here we need a minor modification of this result. For two graphs H_1 and H_2, an (H_1, H_2)-*factor* is any graph that consists only of vertex disjoint copies of H_1 and H_2. The following theorem concerning the appearance of an almost (H_1, H_2)-factors in $G(n, p)$ can be proved with trivial modifications to the proof of [22, Theorem 4.9].

THEOREM 2.3 (Almost factors in $G(n, p)$). *For every pair of graphs* H_1 *and* H_2, *and every* $\varepsilon > 0$ *there is a constant* C *such that, if* $p \geqslant Cn^{-1/m_1(H_i)}$ *for* $i = 1, 2$, *then for every* (H_1, H_2)-*factor* F^* *on at most* $(1-\varepsilon)n$ *vertices, whp* $G(n, p)$ *contains* F^*.

For our result on spanning bounded degree subgraphs, we shall also use the following result of Riordan [36], which allows the embedding of spanning subgraphs that are not locally too dense in $G(n, p)$. For a graph H, let

$$\gamma(H) = \max_{S \subseteq H, v(S) \geqslant 3} \frac{e(S)}{v(S)-2}.$$

Riordan's Theorem can be found in the following form in [34]. We shall use this theorem for a subgraph H of F which excludes the "dense spots" of F.

THEOREM 2.4 (Riordan [36]). *Let* $\Delta \geqslant 2$ *be an integer,* $H \in \mathcal{F}(n, \Delta)$ *and* $p = \omega(n^{-\frac{1}{\gamma(H)}})$. *Then, a copy of* H *is contained in* $G(n, p)$ *whp.*

Finally, we shall use the following submartingale-type inequality to handle weak dependencies in the proof of our main technical result. A proof of this lemma can, for example, be found in [2, Lemma 2.2].

LEMMA 2.5 (Sequential dependence lemma). *Let* Ω *be a finite probability space, and let* $\mathcal{F}_0, \ldots, \mathcal{F}_m$ *be partitions of* Ω, *with* $\mathcal{F}_{i-1}$ *refined by* $\mathcal{F}_i$ *for each* $i \in [m]$. *For each* $i \in [m]$, *let* Y_i *be a Bernoulli random variable on* Ω *which is constant on each part of* $\mathcal{F}_i$. *Let* δ *be a real number,* $\gamma \in (0, 1)$, *and* $X = Y_1 + \cdots + Y_m$. *If* $\mathbb{E}[Y_i | \mathcal{F}_{i-1}] \geqslant \delta$ *holds for all* $i \in [m]$, *then*

$$\mathbb{P}[X \leqslant (1-\gamma)\delta m] \leqslant \exp\left(\frac{-\gamma^2 \delta m}{3}\right).$$

§3. *Main technical theorem.* We start with an outline of the main idea of our strategy for embedding some spanning graph F into $G_\alpha \cup G(n, p)$. Recall that $G(n, p)$ has vertex set $[n]$. We use two-round exposure. In the first round, we will find an F^*-copy for some almost

spanning induced subgraph F^* of F. One key idea in our proof is that, by symmetry, the F^*-copy we find is random among all possible F^*-copies in the complete graph on vertex set $[n]$ (see § 3.1). Hence, it remains to complete such a random F^*-copy to an F-copy using only edges in G_α and the second round (see § 3.2.). It is the additional edges of G_α in this second step that allow us to gain on the bound for embedding F in a random graph alone.

For the second round, we use an absorbing method, relying on the following family of reservoir sets.

Definition 3.1 (Reservoir sets). Given a graph G_α on vertex set $[n]$, a copy $\hat{F}$ of a subgraph F^* of F in the complete graph on vertex set $[n]$ and an independent set W of vertices of $\hat{F}$, we define the family of $(G_\alpha, \hat{F}, W)$-*reservoir sets* $(R(u))_{u\in[n]}$ by setting

$$R(u) = \left\{w \in W : N_{\hat{F}}(w) \subseteq N_{G_\alpha}(u)\right\}. \tag{1}$$

The crucial property of these reservoir sets is as follows. Assume that $\hat{F}$ is a copy of F^* in $G(n, p)$. Then, for any vertex $u \in [n] \setminus V(\hat{F})$ exchanging u with any vertex $w \in R(u)$ gives us a different copy of F^* in $G_\alpha \cup G(n, p)$, now using u. In this case, we also say that we can *switch* u and w. Moreover, since W is an independent set, switching several vertices simultaneously in this manner does not create conflicts. As part of our proof we will show (see Lemma 3.5) that, for a random $\hat{F}$ and a suitably chosen set W, the sets $R(u)$ are likely to have linear size intersections with neighbourhoods in G_α. This will give us "enough room" to complete $\hat{F}$ to F.

Next, we will state the technical embedding theorem, Theorem 3.3, that formalises this method. Theorems 1.4 and 1.3 will be inferred from this result. In our technical theorem, we are given, along with F, a family $\mathcal{F}$ of almost spanning subgraphs of F. This family is chosen such that whp one of these subgraphs appears in our first round and such that in our second round whp each subgraph in $\mathcal{F}$ can be extended to F, using vertex switching. We call a set $\mathcal{F}$ with these properties suitable, defined formally as follows.

Definition 3.2 (Suitability). Let F be an n-vertex graph with maximum degree Δ. A set $\mathcal{F}$ of induced subgraphs of F is called (α, p)-*suitable* if, with

$$\varepsilon = \left(\frac{\alpha}{4\Delta}\right)^{2\Delta}, \tag{2}$$

each graph in $\mathcal{F}$ has at least $(1-\varepsilon)n$ vertices and the following two properties hold.

(A1) $\mathbb{P}(\exists F^* \in \mathcal{F}$ with some F^*-copy in $G(n, p/2)) = 1 - o(1)$.

(A2) Suppose that $F^* \in \mathcal{F}$ and G is a graph with vertex set $[2n]$ which contains a copy $\hat{F}$ of F^*. For each $v \in V(F) \setminus V(F^*)$, let $B(v) \subseteq [2n] \setminus V(\hat{F})$ be a set such that $|B(v) \cap N_G(w)| \geqslant 4\varepsilon n$ for each $w \in [2n]$. Then whp $\hat{F}$ can be extended to a copy of F in $G \cup G(2n, p/6)$ such that each vertex $v \in V(F) \setminus V(F^*)$ is mapped to a vertex in $B(v)$.

Observe that in (A2) we consider auxiliary graphs on $[2n]$. These encode all the information we need from G_α and our second round of randomness. The sets $B(v)$ then are the corresponding auxiliary versions of our reservoir sets. This setup, using $[2n]$, allows us to keep the auxiliary reservoir sets disjoint from the F^*-copy. The idea is, if F^* can be extended to F in this auxiliary graph, then this corresponds to a homomorphism of F in the original setting on $[n]$, and we can use switches to turn this homomorphism into an embedding.

We remark that in the proof of our first result, Theorem 1.4 on squares of Hamilton cycles, the family $\mathcal{F}$ only contains a single graph. In the proof of Theorem 1.3, however, the use of a larger family is crucial.

THEOREM 3.3 (Main technical result). *Let $\alpha > 0$ and $\Delta \in \mathbb{N}$ be constant and let $p = p(n)$. If G_α and F are n-vertex graphs such that*
(i) $V(G_\alpha) = [n]$ *and* $\delta(G_\alpha) \geqslant \alpha n$;
(ii) $\Delta(F) = \Delta$ *and F has an (α, p)-suitable set of subgraphs $\mathcal{F}$;*
then $G_\alpha \cup G(n, p)$ whp contains a copy of F.

The main work for deducing our main results from this theorem will go into finding an (α, p)-suitable family $\mathcal{F}$. Verifying (A1) corresponds to finding an almost spanning embedding for some $F^* \in \mathcal{F}$, which is usually not too hard, because εn vertices remain uncovered. To show (A2), by the definition of the $B(v)$ there is a linear number of options for the embedding of every vertex, which makes this step again be somewhat similar to an almost spanning embedding (and we can also use the edges of G).

We will argue in § 7 that using this theorem we can also easily derive short proofs for a number of related results from the literature. We now turn to the proof of Theorem 3.3.

3.1. *Reducing the problem to completing a random subgraph copy.* In this section we show that, using two-round exposure and (A1), we can reduce the problem of embedding F in $G_\alpha \cup G(n, p)$ to extending a random copy of an almost spanning subgraph.

LEMMA 3.4. *Let α, Δ, p and G_α, F, and $\mathcal{F}$ be as in the hypothesis of Theorem 3.3. For each $F^* \in \mathcal{F}$, let $\hat{F}$ be a random F^*-copy in the complete graph on vertex set $[n]$, and assume*

$$\mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup \hat{F} \cup G(n, p/2)) = 1 - o(1)\,. \tag{3}$$

Then $G_\alpha \cup G(n, p)$ whp contains a copy of F.

Proof. Let G_1 and G_2 be two independent copies of $G(n, p/2)$. For finding a copy of F in $G(n, p)$, we want to use the edges of G_1 to find a copy of $F^* \in \mathcal{F}$, and then use (3) to complete such a copy to F using the edges of G_2 and G_α. For the second step, we will condition on the success of the first step. For this purpose, we define the following events. Let $F_1^*, \dots, F_r^*$ be the graphs in $\mathcal{F}$. For each $1 \leqslant i \leqslant r$, let $\mathcal{E}_i$ be the event that there is a copy of F_i^* in G_1, but no copy of F_j^* for every $j < i$. Note that this event is empty if F_j^* is a subgraph of F_i^* for some $j < i$. These events are chosen such that

$$\sum_{i=1}^{r} \mathbb{P}(\mathcal{E}_i) = \mathbb{P}(\exists i \text{ with some } F_i^*\text{-copy in } G_1) = 1 - o(1)\,, \tag{4}$$

where the second equality uses (A1).

In order to use (3) in the second step, it is essential that we obtain a random copy of $F^* \in \mathcal{F}$ in the first step. Here, the crucial observation is that for each $i \in [r]$ and a random F_i^*-copy $\hat{F}_i$ in the complete graph on vertex set $[n]$, we have

$$\mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup G_1 \cup G_2 | \mathcal{E}_i) \geqslant \mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup \hat{F}_i \cup G_2)\,. \tag{5}$$

Indeed, this follows from the fact that G_1 is independent of $G_\alpha \cup G_2$, and that, if we condition on $\mathcal{E}_i$, then G_1 contains an F_i^*-copy by definition and by symmetry each possible F_i^*-copy is equally likely to appear in G_1. It follows that

$$\mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup G(n,p)) \geqslant \mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup G_1 \cup G_2)$$

$$\geqslant \sum_{i=1}^{r} \mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup G_1 \cup G_2 | \mathcal{E}_i) \cdot \mathbb{P}(\mathcal{E}_i)$$

$$\overset{(5)}{\geqslant} \sum_{i=1}^{r} \mathbb{P}(\exists \text{ an } F\text{-copy in } G_\alpha \cup \hat{F}_i \cup G_2) \cdot \mathbb{P}(\mathcal{E}_i) \overset{(3)}{=} (1-o(1)) \cdot \sum_{i=1}^{r} \mathbb{P}(\mathcal{E}_i) \overset{(4)}{=} 1-o(1)\,,$$

as desired. □

3.2. *Completing a random subgraph copy.* In this section, we provide the proof of our main technical theorem, Theorem 3.3. By Lemma 3.4, it remains to show that whp we can complete a random F^*-copy into a copy of F. For this we will choose a large 2-independent set W in the F^*-copy, which has no neighbours outside F^* (this is with respect to F^* as a subgraph of F), construct the according reservoir sets and perform switches. Recall that a set W of vertices in a graph is called 2-*independent*, if it is independent and no pair of distinct vertices in W have a common neighbour. The following lemma, whose proof we defer to the end of the section, states that these reservoir sets are well distributed with respect to G_α-neighbourhoods.

LEMMA 3.5. *Let α, Δ, p and G_α, F and $\mathcal{F}$ be as in Theorem 3.3. Let $F^* \in \mathcal{F}$ and let W^* be a maximally 2-independent set in F^*, which has no neighbours outside F^*. Let $\hat{F}$ be a random copy of F^* in the complete graph on vertex set $[n]$ and W be the image of W^* in $\hat{F}$. Then whp the $(G_\alpha, \hat{F}, W)$-reservoir sets $(R(u))_{u\in[n]}$ satisfy that for each $u, v \in [n]$, we have*

$$|N_{G_\alpha}(v) \cap R(u)| \geqslant 4\varepsilon n\,,$$

where $\varepsilon = (\frac{\alpha}{4\Delta})^{2\Delta}$, as in (2).

This lemma in particular implies that the sets $R(u)$ are linear in size.

Proof of Theorem 3.3. Assume that we are given graphs G_α and F satisfying the assumptions and a suitable set of almost spanning subgraphs $\mathcal{F}$ of F. Fix $F^* \in \mathcal{F}$ and let $\hat{F}$ be a random copy of F^* in the complete graph on vertex set $[n]$ and let g_0 be the embedding that maps F^* to $\hat{F}$.

By Lemma 3.4, it suffices to prove (3). For this purpose, we will use the reservoir sets and (A2). So, let W^* be a maximally 2-independent set in F^*, which has no neighbours outside F^*, let W be the image of W^* under g_0, and let $(R(u))_{u\in[n]}$ be the $(G_\alpha, \hat{F}, W)$-reservoir sets. By Lemma 3.5, whp, for all $u, v \in [n]$ we have $|N_{G_\alpha}(v) \cap R(u)| \geqslant 4\varepsilon n$.

We now start by mapping the remaining vertices of F arbitrarily to the unused vertices $[n] \setminus V(\hat{F})$. Our goal then is to use switchings to turn this mapping into an embedding of F. So, label the vertices in $[n] \setminus V(\hat{F})$ arbitrarily as $\{z_v : v \in V(F) \setminus V(F^*)\}$. In order to appeal to (A2), we now define an auxiliary graph G on vertex set $[2n]$ together with a collection of auxiliary reservoir sets $B(u)$, which encode the embedding g_0 of F^* and the edges of G_α as well as the reservoir sets $R(u)$.

Let G be the auxiliary graph on the vertex set $[2n]$ that contains all edges of $\hat{F}$ in addition to exactly the following edges. For each edge uw of G_α, the graph G contains the edges $\{u+n, w\}$, $\{u, w+n\}$ and $\{u+n, w+n\}$. For each $v \in V(F) \setminus V(F^*)$, we define the auxiliary reservoir set $B(v) = \{w+n : w \in R(z_v)\}$. Since $|N_{G_\alpha}(v) \cap R(u)| \geqslant 4\varepsilon n$ for all $u, v \in [n]$, we have for

each $v \in V(F) \setminus V(F^*)$ and $w \in [2n]$ that $|B(v) \cap N_G(w)| \geqslant 4\varepsilon n$. So the graph G and the sets $B(v)$ fit the setup in (A2).

Now let G_2' be a copy of $G(2n, p/6)$ on vertex set $[2n]$. Hence, by (A2) the following event $\mathcal{E}$ holds whp: $\hat{F}$ can be extended to a copy of F in $G \cup G_2'$ such that each $v \in V(F) \setminus V(F^*)$ is mapped to $B(v)$. The corresponding embedding g' of F into $G \cup G_2'$ extends g_0. In particular, this F-copy in the auxiliary graph encodes which vertices get switched where (as we detail below).

Now we need to translate this back to our original setting on n vertices. For this, let G_2 be the graph on vertex set $[n]$ and with all edges uw such that $\{u, w+n\}$, $\{u+n, w\}$ or $\{u+n, w+n\}$ is an edge in G_2'. Hence, G_2 is distributed as a random graph in which each edge appears independently and with probability at most $p/2$. Therefore, in order to show (3), it is sufficient to prove that whenever the event $\mathcal{E}$ holds for G_2', then there also is an F-copy in $G_\alpha \cup \hat{F} \cup G_2$.

Indeed, assume that $\mathcal{E}$ holds and define for each $v \in V(F)$

$$g(v) = \begin{cases} g'(v) - n & \text{if } v \in V(F) \setminus V(F^*), \\ z_u & \text{if } g'(v) = g'(u) - n \text{ for some } u \in V(F) \setminus V(F^*), \\ g'(v) & \text{otherwise}. \end{cases}$$

In other words, the first line states that all vertices v in $V(F) \setminus V(F^*)$, which by the definition of $B(v)$ are embedded by g' in $[2n] \setminus [n]$, are mapped by g to the corresponding vertex in $[n]$. The third line guarantees that vertices v in $V(F^*)$ usually are embedded by g' as by g, unless this creates a conflict with the rule from the first line for a vertex u, in which case they are switched to z_u by the second line.

We claim that g is an embedding of F into $G_\alpha \cup \hat{F} \cup G_2$. To see this, let

$$Z_0 = V(F) \setminus V(F^*) \text{ and } Z_1 = \{v : g'(v) = g'(u) - n \text{ for some } u \in Z_0\}.$$

Note that g agrees with g' outside of $Z_0 \cup Z_1$, so that g (appropriately restricted) is an embedding of $F - (Z_0 \cup Z_1)$ into $G_\alpha \cup \hat{F}$. Now consider any $v \in Z_1$ and let $u \in Z_0 = V(F) \setminus V(F^*)$ be such that $g'(v) = g'(u) - n$. Since u is embedded by g' into $B(u) = \{w + n : w \in R(z_u)\}$, we have $g'(v) = g'(u) - n \in R(z_u)$. Recall that $R(z_u) \subseteq W$ by the definition of the reservoir sets, and W is the image under g' of W^*. We conclude that $Z_1 \subseteq W^*$, that is, Z_1 is 2-independent and has no neighbours outside F^*. It follows that vertices in Z_1 have no F-neighbours in Z_0 or Z_1. Thus, for each $v \in Z_1$,

$$g(N_F(v)) = g'(N_F(v)) = N_{\hat{F}}\big(g'(v)\big) \subseteq N_{G_\alpha}(z_u),$$

where the last step uses $g'(v) \in R(z_u)$. This shows that vertices in Z_1 are properly embedded by g.

It remains to consider vertices $v \in Z_0$. We prove that all neighbours of v are mapped to neighbours of $g(v)$, distinguishing three cases. Firstly, for $u \in N_F(v) \setminus (Z_0 \cup Z_1)$, there is an edge between $g(v) = g'(v) - n$ and $g(u) = g'(u)$ in $G_\alpha \cup G_2$, because there is an edge between $g'(v)$ and $g'(u)$ in $G \cup G_2'$. Secondly, for $u \in N_F(v) \cap Z_0$, there is an edge between $g(v) = g'(v) - n$ and $g(u) = g'(u) - n$ in $G_\alpha \cup G_2$, because there is an edge between $g'(v)$ and $g'(u)$ in $G \cup G_2'$. Finally, $N_F(v) \cap Z_1$ is empty, because vertices in Z_1 do not have any F-neighbours in Z_0.

We conclude that g is an embedding of F into $G_\alpha \cup \hat{F} \cup G_2$, completing the proof of Theorem 3.3. □

It remains to prove Lemma 3.5, which is based on the fact that the reservoir sets $R(u)$ are random sets.

Proof of Lemma 3.5. Note that, as F has maximum degree at most Δ, we have

$$|W^*| \geqslant (|F^*| - \Delta|V(F) \setminus V(F^*)|)/\Delta^2 \geqslant n/(2\Delta^2).$$

Let g_0 be the (random) mapping of F^* to $\hat{F}$, and observe that, by symmetry, $W = g_0(W^*)$ is a uniformly random set of size $|W^*|$ in $[n]$.

Fix $u, v \in V(G_\alpha)$. For each $w^* \in W^*$, note that $|N_{F^*}(w^*)| \leqslant \Delta$ and that the sets $\{w^*\} \cup N_{F^*}(w^*)$ are all disjoint. Let I_{w^*} be the indicator variable for the event $g_0(w^*) \in N_{G_\alpha}(v)$ and $N_{\hat{F}}(g_0(w^*)) \subseteq N_{G_\alpha}(u)$. Since by definition $R(u) = \{w \in W : N_{\hat{F}}(w) \subseteq N_{G_\alpha}(u)\} = \{g_0(w^*) : w^* \in W^*, N_{\hat{F}}(g_0(w^*)) \subseteq N_{G_\alpha}(u)\}$, it follows that

$$|N_{G_\alpha}(v) \cap R(u)| = \sum_{w^* \in W^*} I_{w^*}\,. \tag{6}$$

Let $r = \frac{\alpha n}{3\Delta^2} \leqslant |W|$ and pick distinct vertices $w_1^*, \dots, w_r^*$ in W^*. Consider revealing the random copy $\hat{F}$ by, firstly, revealing the mapping of vertices in $\{w_1^*\} \cup N_{F^*}(w_1^*)$, then revealing the mapping of vertices in $\{w_2^*\} \cup N_{F^*}(w_2^*)$, and so on, until $\{w_r^*\} \cup N_{F^*}(w_r^*)$, before finally revealing the rest of the vertices in $\hat{F}$. Note that, for each $1 \leqslant i \leqslant r$, when the location of the vertices in $\{w_i^*\} \cup N_{\hat{F}}(w_i^*)$ is revealed there are at least $\alpha n/2$ vertices both in $N_{G_\alpha}(u)$ and $N_{G_\alpha}(v)$ which are not occupied by a vertex in $\{w_j^*\} \cup N_{F^*}(w_j^*)$ with $j < i$. Hence, for each $1 \leqslant i \leqslant r$, if $m = |N_{F^*}(w_i^*)|$ and $\mathcal{H}_i$ is the history of the location of the vertices in $\{w_j^*\} \cup N_{F^*}(w_j^*)$ with $j < i$, then

$$\mathbb{E}(I_{w_i^*}|\mathcal{H}_i) \geqslant \frac{\alpha n/2 \cdot \binom{(\alpha n/2)-1}{m}}{n\binom{n}{m}} \geqslant \left(\frac{\alpha}{4}\right)^{m+1} \geqslant \left(\frac{\alpha}{4}\right)^{\Delta+1}. \tag{7}$$

Therefore, by (6) and Lemma 2.5 applied with $\delta = (\frac{\alpha}{4})^{\Delta+1}$, we have

$$|N_{G_\alpha}(v) \cap R(u)| \geqslant 3\delta r/4 \geqslant \frac{\alpha^{\Delta+2} n}{4^{\Delta+2}\Delta^2} \geqslant 4\Big(\frac{\alpha}{4\Delta}\Big)^{2\Delta} n = 4\varepsilon n\,,$$

with probability $1 - \exp(-\Omega(\delta r)) = 1 - o(n^{-2})$. Using a union bound, we conclude that with probability $1 - o(1)$ for each $u, v \in V(G_\alpha)$, we have $|N_{G_\alpha}(v) \cap R(u)| \geqslant 4\varepsilon n$. □

§4. *Powers of Hamilton cycles.* Let $F = C_n^{(k)}$ be the kth power of the cycle with n vertices, and let $P_n^{(k)}$ denote the kth power of a path with n vertices. To prove Theorem 1.4, it is sufficient, by Theorem 3.3, to find an $\eta = \eta(\alpha) > 0$, such that there exists an (α, p)-suitable set $\mathcal{F}$ of subgraphs of F with

$$p = n^{-1/k-\eta}\,.$$

In fact, we will use only one subgraph, which will consist of disjoint copies of the kth power of long (but constant length) paths, which we connect by shorter kth powers of paths to form a copy of F.

In the following, we shall explain how we choose $\mathcal{F}$, and show that $\mathcal{F}$ satisfies (A1) and (A2) for $p = n^{-1/k-\eta}$, which implies that $\mathcal{F}$ is (α, p)-suitable. We use the following constants. Given k and $\alpha > 0$, let $\Delta = 2k$ and $\varepsilon = (\frac{\alpha}{4\Delta})^{2\Delta}$. Pick large integers m and ℓ, and a small constant $\eta > 0$ such that

$$\alpha, \frac{1}{k} \gg \frac{1}{\ell} \gg \frac{1}{m} \gg \eta > 0\,,$$

where, for example, by $\frac{1}{m} \gg \eta$ we mean that the following proof works if we choose η sufficiently small compared to $1/m$. In particular, we require $\ell^2 \leqslant \varepsilon m$.

4.1. *Choosing* $\mathcal{F}$. Let $\mathcal{F}$ solely contain F^*, the following $(P_m^{(k)}, P_{m+1}^{(k)})$-factor on at least $(1-\varepsilon)n$ vertices, which is a subgraph of F. Let s and t be the unique integers such that $n = s(m+\ell)+t$ and $t < (m+\ell)$. Let F^* be the graph on $v(F^*) = sm + t = t(m+1) + (s-t)m$ vertices consisting of the following vertex disjoint kth powers of paths: t copies of $P_{m+1}^{(k)}$, which we denote by $P_1^*, \dots, P_t^*$, and $s-t$ copies of $P_m^{(k)}$, which we denote by $P_{t+1}^*, \dots, P_s^*$. This leaves exactly $v(F) - v(F^*) = s\ell \leqslant s\varepsilon m \leqslant \varepsilon n$ vertices of F uncovered.

Observe that we obtain F from F^* by connecting for each $i \in [s]$ the paths P_i^* and P_{i+1}^* (respectively, P_1^* if $i = s$) by a kth power of a path with ℓ vertices, which we denote by $w_{i,1}^*, \dots, w_{i,\ell}^*$, such that the following is satisfied. For $i \in [s]$, let $u_{i,1}^*, \dots, u_{i,k}^*$ be the end k-tuple of P_i^* and $v_{i,1}^*, \dots, v_{i,k}^*$ be the start k-tuple of P_{i+1}^* (respectively, P_1^* if $i = s$). We require that

$$u_{i,1}^*, \dots, u_{i,k}^*, w_{i,1}^*, \dots, w_{i,\ell}^*, v_{i,1}^*, \dots, v_{i,k}^*$$

is the kth power of a path with $\ell + 2k$ vertices.

4.2. *Proof that* $\mathcal{F}$ *satisfies* (A1). We use Theorem 2.3 to find a copy of F^* in $G(n, p/2)$. Since for $m' \geqslant 2k$, we have $e(P_{m'}^{(k)}) = km' - \binom{k+1}{2}$, it is easy to check that for $k \geqslant 2$ we have $m_1(P_m^{(k)}), m_1(P_{m+1}^{(k)}) < k$. Since F^* is an $(P_m^{(k)}, P_{m+1}^{(k)})$-factor on at most $(1-\varepsilon)n$ vertices, it follows directly from Theorem 2.3 that $G(n, p/2)$ contains a copy of F^*, and hence (A1) holds for $\mathcal{F}$.

4.3. *Proof that* $\mathcal{F}$ *satisfies* (A2). Suppose that G is a graph with vertex set $[2n]$ which contains a copy $\hat{F}$ of F^*. For each $v \in V(F) \setminus V(F^*)$, assume we are given a set $B(v) \subseteq [2n] \setminus V(\hat{F})$ such that for each $w \in [2n]$ we have $|B(v) \cap N_G(w)| \geqslant 4\varepsilon n$. Let $G' = G(2n, p/6)$. Our goal is to extend $\hat{F}$ to a copy of F in $G \cup G'$ such that each vertex v in $V(F) \setminus V(F^*)$ is mapped to $B(v)$.

For each $i \in [s]$ and $j \in [k]$, let $u_{i,j}$ be the image of $u_{i,j}^*$ in $\hat{F}$, and $v_{i,j}$ be the image of $v_{i,j}^*$ in $\hat{F}$. Hence, to extend $\hat{F}$ to a copy of F we need to embed all vertices $w_{i,j}^*$ with $i \in [s]$ and $j \in [\ell]$ to distinct vertices $w_{i,j}$ so that

$$u_{i,1}, \dots, u_{i,k}, w_{i,1}, \dots, w_{i,\ell}, v_{i,1}, \dots, v_{i,k} \tag{8}$$

is the kth power of a path with $2k + \ell$ vertices.

We would like to appeal to Hall's condition for hypergraphs, Theorem 2.1, to show that this is possible. For this purpose, we define the following auxiliary hypergraphs. Let $W = [2n] \setminus V(\hat{F})$. For each $i \in [s]$, let L_i be the ℓ-uniform hypergraph with vertex set W where $e \in \binom{W}{\ell}$ is an edge exactly if there is some ordering of e as $w_{i,1}, \dots, w_{i,\ell}$ so that (8) is the kth power of a path in $G \cup G'$ and $w_{i,j} \in B(w_{i,j}^*)$ for each $j \in [\ell]$. We shall argue that the following lemma, whose proof we defer to §4.4, guarantees that the assumption of Theorem 2.1 is satisfied.

LEMMA 4.1. *For each* $r \in [s]$ *and* $A \subseteq [s]$ *with* $|A| = r$ *and* $U \subseteq W$ *with* $|U| \leqslant \ell^2 r$, *the following holds with probability at least* $1 - \exp(-\omega(r \log n))$. *There exists some* $i \in A$ *and an edge* $e \in E(L_i)$ *with* $V(e) \subseteq W \setminus U$.

The property in Lemma 4.1 fails for some $r \in [s]$ and $A \subseteq [s]$ with $|A| = r$ and $U \subseteq W$ with $|U| \leqslant \ell^2 r$ with probability at most

$$\sum_{r \in [s]} \binom{s}{r} \binom{2n}{\ell^2 r} \cdot \exp(-\omega(r \log n)) = o(1),$$

so we may assume the property holds for all such sets.

To apply Theorem 2.1, we need to show that, for every $A \subseteq [s]$, the hypergraph $\bigcup_{i \in A} L_i$ contains a matching with size greater than $\ell(|A| - 1)$. Indeed, let $A \subseteq [s]$ and $r = |A|$, and let U be the vertex set of a maximal matching in $\bigcup_{i \in A} L_i$. This means that there is no $i \in A$ and edge $e \in E(L_i)$ with $V(e) \subseteq W \setminus U$. Thus, by the property from Lemma 4.1, we have $|U| \geqslant \ell^2 |A|$, so that $\bigcup_{i \in A} L_i$ contains a matching with size at least $\ell |A|$. Therefore, we can apply Theorem 2.1, and obtain a function $\pi : [s] \to \bigcup_{i \in [s]} E(L_i)$ such that $\pi(i) \in E(L_i)$ for each $i \in [s]$ and the edges in $\pi([s])$ are vertex disjoint. Observe that, by the definition of the hypergraphs L_i, embedding the vertices $w^*_{i,1}, \dots, w^*_{i,\ell}$ to the vertices of $\pi(i)$ in an appropriate order yields the desired completion of $\hat{F}$ to an embedding of F. Thus, subject only to the proof of Lemma 4.1, (A2) holds as required.

4.4. *Proof of Lemma 4.1.* We will prove Lemma 4.1 using Janson's inequality, Lemma 2.2. Recall that the hyperedges of each hypergraph L_i represent legitimate connections in $G \cup G'$ between the images of the kth power of paths P^*_i and P^*_{i+1} in $\hat{F}$.

Proof of Lemma 4.1. Fix $r \in [s]$ and $A \subseteq [s]$, $U \subseteq W$ with $|A| = r$ and $|U| \leqslant \ell^2 r \leqslant \ell^2 s \leqslant \varepsilon n$. Let $j = \lfloor \ell/2 \rfloor$. Let P be the kth power of the path with vertex set

$$u_1, \dots, u_k, w_1, \dots, w_\ell, v_1, \dots, v_k, \tag{9}$$

with all the edges between the vertices u_i removed and all the edges between the vertices v_i removed. Furthermore, remove from P the edges $u_k w_1$, $w_\ell v_1$ and all the edges $w_i w_{i+1}$, $i \in [\ell - 1]$, except for $w_j w_{j+1}$. The edges that we have removed will come from the deterministic graph G, while we will find a copy of P in G'. The edge $w_j w_{j+1}$ is included in P so that we do not need to find a path between v_k and w_1 in G.

To simplify our calculations for the application of Janson's inequality, let us first prove three simple claims concerning the density of subgraphs of P. Let $U = \{u_1, \dots, u_k\}$ and $V = \{v_1, \dots, v_k\}$.

CLAIM 4.2. $e(P) \leqslant \ell(k - 1/2)$.

Proof of Claim 4.2. In the ordering of the vertices in P in (9), ignoring the edge $w_j w_{j+1}$, each vertex has at most $k - 1$ neighbours to the right. Therefore, including the edge $w_j w_{j+1}$, we have $e(P) \leqslant (\ell + 2k)(k - 1) + 1 \leqslant \ell(k - 1/2)$, since we chose $\ell \gg k$. □

CLAIM 4.3. *For each subgraph $P' \subseteq P - (U \cup V)$ with $e(P') \geqslant 1$, we have $p^{-e(P')} \cdot n^{1 - v(P')} = o(\log^{-1} n)$.*

Proof of Claim 4.3. Removing the edge $w_j w_{j+1}$ if necessary, we have that each vertex in P' has at most $(k - 1)$ neighbours to the right in the labelling in (9). As the rightmost vertex in P' has no such neighbours, if $v(P') \geqslant 3$, then we have $e(P') \leqslant (k - 1)(v(P') -$

$1) + 1 \leqslant k(v(P') - 1) - 1$. If $v(P') = 2$, then $e(P') = 1 \leqslant k(v(P') - 1) - 1$. Therefore, as $\eta \ll 1/\ell, 1/k$,

$$p^{-e(P')} \cdot n^{1-v(P')} \leqslant p(p^k n)^{1-v(P')} = n^{-1/k-\eta}(n^{-k\eta})^{1-v(P')} \leqslant n^{-1/k-\eta+k\ell\eta} = o(\log^{-1} n).$$

□

CLAIM 4.4. *For each subgraph $P' \subseteq P$ with $P' \neq P$, $e(P') \geqslant 1$ and $U, V \subseteq V(P')$, we have $p^{-e(P')} \cdot n^{2k-v(P')} = o(\log^{-1} n)$.*

Proof of Claim 4.4. For such a subgraph P', let $W_0 = V(P') \setminus (U \cup V)$. We enumerate the vertices from W_0 by $w_{i_1}, \ldots, w_{i_t}$ from left to right in the ordering (9). If there is an index a with $i_{a+1} - i_a \geqslant k + 1$, then we estimate the number of edges in P' through $(k-1)|W_0| + 1$. This is so because we can enumerate all the edges of P' by identifying at least one vertex adjacent to every edge of P' as follows: every vertex w_{i_c} $(c \leqslant a)$ is adjacent to the left to at most $k-1$ vertices, and every vertex w_{i_c} $(c > a)$ is adjacent to the right to at most $k-1$ vertices, the only exception being possibly the vertices w_j and w_{j+1} along with the edge $w_j w_{j+1}$, thus contributing one more possible edge. Therefore, if $|W_0| \geqslant 2$, then

$$e(P') \leqslant (k-1)|W_0| + 1 \leqslant k|W_0| - 1 \leqslant k(v(P') - 2k) - 1.$$

If $|W_0| = 1$, then, as $\ell \gg k$, the vertex in W_0 cannot have neighbours in both U and V, so that $e(P') \leqslant (k-1) = k(v(P') - 2k) - 1$.

If there is no such index a as above, then note that $v(P') \geqslant |W_0| \geqslant (\ell - k)/k \gg k^2$. Then, counting from the edges of P' from their leftmost vertex in (9), and remembering that $w_j w_{j+1}$ may be an edge, $e(P') \leqslant (k-1)v(P') + 1 \leqslant k(v(P') - 2k) + 2k^2 - v(P') + 1 \leqslant k(v(P') - 2k) - 1$. Thus, in all cases, $e(P') \leqslant k(v(P') - 2k) - 1$.

Therefore, as $\eta \ll 1/\ell$,

$$p^{-e(P')} \cdot n^{2k-v(P')} \leqslant p(p^k n)^{2k-v(P')} = n^{-1/k-\eta}(n^{-k\eta})^{2k-v(P')} \leqslant n^{-1/k-\eta+k\ell\eta} = o(\log^{-1} n).$$

□

For each $i \in [A]$, let $\mathcal{P}_i$ be the set of copies of P in the graph G' with vertices in order (to match (9))

$$u_{i,1}, \ldots, u_{i,k}, w_{i,1}, \ldots, w_{i,\ell}, v_{i,1}, \ldots, v_{i,k},$$

where $w_{i,1} \in B(w^*_{i,1}) \cap N_G(u_{i,k})$, $w_{i,j'+1} \in B(w^*_{i,j'+1}) \cap N_G(w_{i,j'})$ for each $j' \in [j-1]$, $w_{i,\ell} \in B(w^*_{i,\ell}) \cap N_G(v_{i,1})$ and $w_{i,j'-1} \in B(w^*_{i,j'-1}) \cap N_G(w_{i,j'})$ for each $j' \in \{j+1, \ldots, \ell\}$. That is, if such a copy of P exists in G', then the edge $\{w_{i,1}, \ldots, w_{i,\ell}\}$ is in L_i.

Note that, choosing the vertices in order $w_{i,1}, \ldots, w_{i,j}, w_{i,\ell}, w_{i,\ell-1}, \ldots, w_{i,j+1}$, there are at least $4\varepsilon n - |U| - s \geqslant 2\varepsilon n$ options for each vertex, and therefore $|\mathcal{P}_i| = \Omega(n^\ell)$. Let $\mathcal{P} = \cup_{i \in A} \mathcal{P}_i$, so that $|\mathcal{P}| = \Omega(r \cdot n^\ell)$.

For each $Q, Q' \in \mathcal{P}$, with $Q \neq Q'$, let $Q \sim Q'$ if Q and Q' share some edge. Let $q = p/6$, the edge probability in G'. Denote the expectation for the number of graphs from $\mathcal{P}$ in G' by $\mu = |\mathcal{P}|q^{e(P)}$ and let

$$\delta = \sum_{Q, Q' \in \mathcal{P}: Q \sim Q'} q^{2e(P) - e(Q \cap Q')}.$$

Note that, as $\eta \ll 1/k$ and $p = n^{-1/k-\eta}$, we have $q^{(k-1/2)}n = \omega(\log n)$. As $|\mathcal{P}| = \Omega(r \cdot n^\ell)$, we then have, using Claim 4.2,

$$\mu = \Omega(r \cdot q^{e(P)} n^\ell) = \Omega(r \cdot (q^{(k-1/2)} n)^\ell) = \omega(r \cdot \log n).$$

Recall that for $j \in [k]$ the vertices $u_{i,j}$ and $v_{i,j}$ denote the images of the end-k-tuples of the graph P_i^* from F^* given through the copy $\hat{F}$ of F^* and moreover that all these k-tuples contain distinct vertices (cf. §4.3). Let $i \in A$ and $Q \in \mathcal{P}_i$, and let $U_i = \{u_{i,1}, \dots, u_{i,k}\}$ and $V_i = \{v_{i,1}, \dots, v_{i,k}\}$. For each $P' \subsetneq Q$ with $U_i, V_i \subseteq V(P')$, there are at most $n^{\ell+2k-v(P')}$ graphs $Q' \in \mathcal{P}$ with $Q \cap Q' = P'$ (all of which are in $\mathcal{P}_i$). For each subgraph $P' \subseteq Q - (U_i \cup V_i)$, there are at most $r \cdot n^{\ell-v(P')} \leqslant n^{\ell+1-v(P')}$ graphs $Q' \in \mathcal{P}$ with $Q \cap Q' = P'$. Thus,

$$\delta \leqslant |\mathcal{P}| \cdot \left(\sum_{P' \subseteq P-(U\cup V):\, e(P') \geqslant 1} q^{2e(P)-e(P')} n^{\ell+1-v(P')} + \sum_{P' \subsetneq P: U, V \subseteq V(P')} q^{2e(P)-e(P')} n^{\ell+2k-v(P')} \right).$$

Therefore, as $|\mathcal{P}| = O(rn^\ell)$ and $\mu = \Omega(rq^{e(P)}n^\ell)$,

$$\frac{\delta r}{\mu^2} = O\left(\sum_{P' \subseteq P-(U\cup V):e(P') \geqslant 1} q^{-e(P')} \cdot n^{1-v(P')} + \sum_{P' \subsetneq P: U, V \subseteq V(P')} q^{-e(P')} n^{2k-v(P')} \right) = o(\log^{-1} n),$$

using Claims 4.3 and 4.4. Thus, as $\mu = \omega(r \log n)$ and $\frac{\delta}{\mu^2} = o(r^{-1} \log^{-1} n)$, we can infer from Janson's inequality, Lemma 2.2, that with probability at least $1 - \exp(-\omega(r \log n))$ there is some $i \in A$ and $Q \in \mathcal{P}_i$ in G', and hence $V(Q) \setminus (U_i \cup V_i) \in E(L_i)$, as required. □

§5. *Spanning subgraphs with bounded maximum degree.* Let $F \in \mathcal{F}(n, \Delta)$ and $p = \omega(n^{-\frac{2}{\Delta+1}})$. As before, we find a suitable set $\mathcal{F}$ of large subgraphs of F such that we can whp embed one of these subgraphs $F^* \in \mathcal{F}$ in $G(n, p/2)$ ((A1) in Definition 3.2), and then extend any such F^*-copy (in an auxiliary graph) to cover all of F ((A2) in Definition 3.2). To do this, we adapt the strategy of Ferber *et al.* [17] to decompose F. In [17], each graph $F \in \mathcal{F}(n, \Delta)$ is decomposed into a sparse part and many dense spots. Our set $\mathcal{F}$ will consist of subgraphs of F covering the sparse part and most of the dense parts.

Recall that the parameter

$$\gamma(H) = \max_{S \subseteq H, v(S) \geqslant 3} \frac{e(S)}{v(S) - 2}$$

determines when we can apply Riordan's theorem, Theorem 2.4, to embed a spanning subgraph in $G(n, p)$. In the following, we call a graph H *dense* if $\gamma(H) > \frac{\Delta+1}{2}$ and *sparse* otherwise. We can now define, following [17], a good decomposition of a graph.

Definition 5.1 (ε-good decomposition)). Let $\varepsilon > 0$, $F \in \mathcal{F}(n, \Delta)$ and let $\mathcal{S}_1, \dots, \mathcal{S}_k$ be families of induced subgraphs of F. For $F' = F - (\bigcup_h \bigcup_{S \in \mathcal{S}_h} V(S))$, we say that $(F', \mathcal{S}_1, \dots, \mathcal{S}_k)$ is an *ε-good decomposition* if the following hold.

(P1) F' is sparse, that is, $\gamma(F') \leqslant \frac{\Delta+1}{2}$.

(P2) Each $S \in \bigcup_h \mathcal{S}_h$ is minimally dense, that is, $\gamma(S) > \frac{\Delta+1}{2}$ and S' is sparse for all $S' \subseteq S$ with $3 \leqslant v(S') < v(S)$.

(P3) For each $1 \leqslant h \leqslant k$, all the graphs in $\mathcal{S}_h$ are isomorphic.

(P4) Every $\mathcal{S}_h$ contains graphs on at most εn vertices, that is $|\bigcup_{S \in \mathcal{S}_h} V(S)| \leqslant \varepsilon n$.

(P5) All the graphs in $\bigcup_i \mathcal{S}_i$ are vertex disjoint and, for each $1 \leqslant h \leqslant k$ and $S, S' \in \mathcal{S}_h$ with $S \neq S'$, there are no edges between S and S' in F, and S and S' share no neighbours in F.

We call the graphs in $\mathcal{S}_1, \dots, \mathcal{S}_k$ the *dense spots* of the decomposition.

We remark that our definition is slightly less restrictive than that from [17], where $(P3)$ is replaced by a stronger condition. An ε-good decomposition can easily be found using a greedy algorithm. The following lemma is proved in [17].

LEMMA 5.2 [17, Lemma 2.2]. *For each $\varepsilon > 0$ and $\Delta > 0$, there exists some k_0 such that, for each $F \in \mathcal{F}(n, \Delta)$, there is some $k \leqslant k_0$ and an ε-good decomposition $(F', \mathcal{S}_1, \dots, \mathcal{S}_k)$ of F.*

In the following, we shall use this lemma to define a family $\mathcal{F}$ of subgraphs of $F \in \mathcal{F}(n, \Delta)$. We shall then show that this family $\mathcal{F}$ satisfies (A1) and (A2) and hence is (α, p)-suitable, which by Theorem 3.3 implies Theorem 1.3 as desired.

5.1. *Choosing $\mathcal{F}$.* Fix $F \in \mathcal{F}(n, \Delta)$. Let $\varepsilon = (\frac{\alpha}{4\Delta})^{2\Delta}$, and let k_0 be large enough for the result of Lemma 5.2 to hold with ε and Δ. By Lemma 5.2, for some $k \leqslant k_0$, there is an ε-good decomposition $(F', \mathcal{S}_1, \dots, \mathcal{S}_k)$ of F, which we fix.

For each $1 \leqslant h \leqslant k$, let s_h be the size of the graphs in $\mathcal{S}_h$ (possible by (P3)), and, picking some representative $S \in \mathcal{S}_h$, note that, by (P2) and as $\Delta(S) \leqslant \Delta$, we have

$$(\Delta + 1)(s_h - 2) < 2e(S) \leqslant \Delta s_h,$$

so that $s_h < 2\Delta + 2$. Thus, we may consider α, Δ, ε, $k \leqslant k_0$ and the maximum size of each dense spot $(2\Delta + 1)$ to be constant, while n tends to infinity.

Let $\mathcal{F}$ contain exactly those induced subgraphs of F which cover F' and, for each $1 \leqslant h \leqslant k$, all but at most $\frac{\varepsilon n}{s_h^2 k}$ of the graphs from $\mathcal{S}_h$.

5.2. *Proof that $\mathcal{F}$ satisfies* (A1). We shall embed the copy of F' using Riordan's theorem, Theorem 2.4. In [17], the embedding of F' is then extended step by step to include the graphs in $\mathcal{S}_h$, for $1 \leqslant h \leqslant k$. We proceed similarly, but in each step only include most of the graphs $\mathcal{S}_h$, for $1 \leqslant h \leqslant k$. This allows us to work at a lower probability than that used in [17], as we aim to find a copy of only some graph in $\mathcal{F}$.

To find such a copy of a graph in $\mathcal{F}$, we expose the graph $G(n, p/2)$ in a total of $k + 1$ rounds, revealing $G_h \sim G(n, q)$ for $0 \leqslant h \leqslant k$, where $q = p/(6k)$ and thus $(1 - q)^{k+1} \geqslant 1 - p/2$. Every edge is thus present with probability at most $p/2$ in $\bigcup_h G_h$. We use G_0 to embed F' and then iteratively use $G_1, \dots, G_k$ to embed as many subgraphs from $\mathcal{S}_1, \dots, \mathcal{S}_k$ as possible, and show that this results whp in an embedding of a subgraph from $\mathcal{F}$.

Since, by (P1), $\gamma(F') \leqslant \frac{\Delta+1}{2}$, and thus $q = \omega(n^{-\frac{1}{\gamma(F')}})$, by Theorem 2.4, we can whp embed F' into G_0. Let $f_0 : V(F') \to V(G_0)$ be such an embedding and let $F_0' = f_0(F')$.

For $1 \leqslant h \leqslant k$, we want to (whp) use edges from G_h to extend the embedding f_{h-1} to cover all but at most $\frac{\varepsilon n}{s_h^2 k}$ graphs from $\mathcal{S}_h$. We then let f_h be the extended embedding and let F_h' be the subgraph of F embedded by f_h. We use the following lemma, which allows us to extend the current embedding to one more dense spot $S \in \mathcal{S}_h$, even if we restrict its image to a small but linearly sized set U, using only edges of G_h. This lemma is proved along with another lemma from this section in § 6.

LEMMA 5.3. *For each $1 \leqslant h \leqslant k$, the following holds whp for any $\mathcal{S} \subseteq \mathcal{S}_h$ and $U \subseteq V(G_\alpha)$ with $|\mathcal{S}| \geqslant \frac{\varepsilon n}{s_h^2 k}$ and $|U| \geqslant \frac{\varepsilon n}{s_h k}$. There is some $S \in \mathcal{S}$ and a copy S' of S in $G_h[U]$ with an embedding $\pi : V(S) \to V(S')$ such that, for each $v \in V(S)$,*

$$f_{h-1}(N_F(v) \cap V(F_{h-1}')) \subseteq N_{G_h}(\pi(v)). \tag{10}$$

Start with f_0 and F_0'. For each $1 \leqslant h \leqslant k$, we construct f_h and F_h', as follows. The property in Lemma 5.3 whp holds for h. We extend the embedding f_{h-1} to f_h using edges from G_h to cover as many of the graphs in $\mathcal{S}_h$ as possible (with any edges to F_{h-1}' correctly embedded), and call the resulting graph F_h'. By the property in Lemma 5.3, this leaves at most $\frac{\varepsilon n}{s_h^2 k}$ graphs in $\mathcal{S}_h$ unembedded. Indeed, if there is a set $\mathcal{S}$ of at least $\frac{\varepsilon n}{s_h^2 k}$ unembedded graphs in $\mathcal{S}_h$, then, let $U = V(G_\alpha) \setminus V(F_h')$ and note that $|U| \geqslant s_h \cdot |\mathcal{S}| \geqslant \frac{\varepsilon n}{s_h k}$. There then exists some $S \in \mathcal{S}$ and a copy S' of S in $G_h[U]$ with isomorphism $\pi : V(S) \to V(S')$ such that (10) holds for each $v \in V(S)$. As, by (P5), no two subgraphs in $\mathcal{S}_h$ have an edge between them, π can be used to embed S and extend the embedding f_h, a contradiction.

From this, we obtain (whp) the embedding f_k of a subgraph of F, covering F' and all but at most $\frac{\varepsilon n}{s_h^2 k}$ graphs from each $\mathcal{S}_h$, $1 \leqslant h \leqslant k$, into $\bigcup_h G_h$. Such a subgraph embedded by f_k is thus in $\mathcal{F}$, and therefore (A1) holds.

5.3. *Proof that $\mathcal{F}$ satisfies* (A2). Let $F^* \in \mathcal{F}$ and let the graph G be as described in (A2) containing the copy $\hat{F}$ of F^*.

For each $1 \leqslant h \leqslant k$, let $\mathcal{S}_h' \subseteq \mathcal{S}_h$ be those dense parts not in F^*, so that $|\mathcal{S}_h'| \leqslant \frac{\varepsilon n}{s_h^2 k}$. We have, for each $1 \leqslant h \leqslant k$, that the graphs in $\mathcal{S}_h'$ are isomorphic, minimally dense, disjoint and neither have edges between them nor share any neighbours. Furthermore, the sets in $\{V(F^*)\} \cup \{V(S) : S \in \mathcal{S}_h', 1 \leqslant h \leqslant k\}$ form a partition of $V(F)$. Note that $|V(F) \setminus V(F^*)| \leqslant \varepsilon n$. For each $0 \leqslant h \leqslant k$, let F_h be the induced subgraph of F with vertex set $V(F^*) \cup (\cup_{h' \leqslant h} \cup_{S \in \mathcal{S}_{h'}} V(S))$.

Let $G_1', \dots, G_k'$ be independent random graphs with $G_i' \sim G(2n, q)$, where $q = p/(6k)$. Starting with g_0 and $F_0 = F^*$, for each $1 \leqslant h \leqslant k$ in turn, we will (whp) inductively find a function

$$g_h : V(F_h) \to [2n]$$

such that

(Q1) g_h is an embedding of F_h into $G \cup (\bigcup_{h' \leqslant h} G_{h'}')$, which extends g_{h-1} and

(Q2) for each vertex $v \in F_h \setminus F_{h-1}$, we have $g_h(v) \in B(v)$.

Note that g_0 satisfies these properties, and that, once we find g_k whp, we will have an embedding of $F_k = F$ into $G \cup (\bigcup_{1 \leqslant h \leqslant k} G_h')$, satisfying the conditions in (A2). Noting that each edge in $\bigcup_h G_h$ appears independently at random with probability at most $p/6$, we then have that (A2) holds.

Suppose then that $1 \leqslant h \leqslant k$ and we have found the function g_{h-1} satisfying (Q1) and (Q2). Let $W_{h-1} = [2n] \setminus g_{h-1}(F_{h-1})$. For each $S \in \mathcal{S}_h'$, label $V(S) = \{z_{S,1}, \dots, z_{S,s_h}\}$, and let L_S be the s_h-uniform auxiliary hypergraph with vertex set W_{h-1}, where e is an edge of L_S if, for some labelling $e = \{w_{S,1}, \dots, w_{S,s_h}\}$, the map $z_{S,i} \mapsto w_{S,i}$ is an embedding of S into $G \cup G_h'$, where, for each $1 \leqslant i \leqslant s_h$ we have $w_{S,i} \in B(z_{S,i})$ and $g_{h-1}(N_{F_h}(z_{S,i}) \cap (V(F_{h-1}))) \subseteq N_{G \cup G_h'}(w_{S,i})$. Each hyperedge $e = \{w_{S,1}, \dots, w_{S,s_h}\}$ of L_S then corresponds to a possible extension of g_{h-1} to cover $S \in \mathcal{S}_h'$.

We wish to show that whp there exists a function $\pi : \mathcal{S}_h' \mapsto \bigcup_{S \in \mathcal{S}_h'} E(L_S)$ such that $\pi(S) \in E(L_S)$ for each $S \in \mathcal{S}_h'$, and the edges in $\pi(\mathcal{S}_h')$ are pairwise vertex disjoint. This is possible, as shown below, using Theorem 2.1 and the following lemma.

LEMMA 5.4. *For each $1 \leqslant h \leqslant k$, $1 \leqslant r \leqslant |\mathcal{S}_h'|$, $\mathcal{S} \subseteq \mathcal{S}_h'$ and $U \subseteq W_{h-1}$, with $|\mathcal{S}| = r$ and $|U| \leqslant s_h^2 r$, the following holds with probability at least $1 - \exp(-\omega(r \log(\frac{n}{r})))$. There exists some $S \in \mathcal{S}$ and an edge $e \in E(L_S)$ with $V(e) \subseteq W_{h-1} \setminus U$.*

The property in Lemma 5.4 then holds for each $1 \leqslant h \leqslant k$, $1 \leqslant r \leqslant |\mathcal{S}'_h|$, $\mathcal{S} \subseteq \mathcal{S}'_h$ and $U \subseteq W_{h-1}$, with $|\mathcal{S}| = r$ and $|U| \leqslant s_h^2 r$ with probability at least

$$1 - k \cdot n \cdot \sum_{r=1}^{|\mathcal{S}'_h|} \binom{n}{r} \cdot \binom{n}{s_h^2 r} \cdot \exp(-\omega(r \log(\tfrac{n}{r}))) = 1 - o(1).$$

Similarly to our deductions from Lemma 4.1, it then follows that, for every $\mathcal{S} \subseteq \mathcal{S}'_h$, the hypergraph $\bigcup_{S \in \mathcal{S}} L_S$ contains a matching with size greater than $s_h(|\mathcal{S}| - 1)$. Therefore, by Theorem 2.1, a function π as described above exists. Thus, we can extend g_{h-1} to an embedding g_h of F_h satisfying (Q1) and (Q2) as required.

Subject to the proof of Lemma 5.4, this completes the proof that (A2) holds.

§6. *Proofs of auxiliary lemmas.*

In this section, we give the proofs of the lemmas from § 5.

6.1. *Proof of Lemma 5.3.*

We prove Lemma 5.3 with Janson's inequality, using similar calculations to Ferber *et al.* [17].

Proof of Lemma 5.3. Fixing h, note that there are certainly at most $2^n \cdot 2^n$ choices for $\mathcal{S}$ and U. Therefore, it is sufficient to prove, for fixed $\mathcal{S} \subseteq \mathcal{S}_h$ and $U \subseteq V(G_\alpha) \setminus f_{h-1}(V(F'_{h-1}))$ with $|\mathcal{S}| \geqslant \frac{\varepsilon n}{s_h^2 k}$ and $|U| \geqslant \frac{\varepsilon n}{s_h k}$, the property in the lemma holds with probability $1 - e^{-\omega(n)}$.

Let $s = s_h$. Pick some $S_0 \in \mathcal{S}$, so that, by (P3), each graph in $\mathcal{S}$ is isomorphic to S_0, and label $V(S_0) = \{v_1, \dots, v_s\}$. Let $\mathcal{H}$ be a set of $\binom{|U|}{s}$ copies of S_0 in the complete graph with vertex set U, where each copy of S_0 has a different vertex set. Note that $|U| = \Omega(n)$ and $|\mathcal{H}| = \Omega(n^s)$. For each $S \in \mathcal{S}$ and $H \in \mathcal{H}$, label $V(S) = \{z_{S,1}, \dots, z_{S,s}\}$ and $V(H) = \{v_{H,1}, \dots, v_{H,s}\}$ so that $v_i \mapsto z_{S,i}$ and $v_i \mapsto v_{H,i}$ are embeddings of S_0.

Each graph in $\mathcal{S}$ is isomorphic to S_0 in F, but, when we come to extend an embedding of F'_{h-1} to F'_h by embedding "most" of the copies from $\mathcal{S}_h$, the number of edges between a copy from $\mathcal{S}_h$ and the already embedded F'_{h-1} may differ. We now distinguish two cases: Case I where each copy S from $\mathcal{S}$ has some edge between S and F'_{h-1} in F'_h and Case II where there is some copy S from $\mathcal{S}$ for which there is no such edge.

Let us assume first that we are in Case I. For each $S \in \mathcal{S}$, let $W_S = f_{h-1}(\bigcup_{v \in V(S)} N_F(v) \cap V(F'_{h-1}))$ be the images of the already embedded neighbours of vertices in S. Note that these sets W_S are nonempty by the definition of Case I and by (P5) are disjoint. For each $H \in \mathcal{H}$ and $S \in \mathcal{S}$, let $H \oplus W_S$ be the graph with vertex set $V(H) \cup W_S$ containing exactly those edges that we need in order to extend the partial embedding we have to embed S into H. That is, $H \oplus W_S$ has edge set

$$E(H) \cup \{v_{H,i}v : 1 \leqslant i \leqslant s, v \in f_{h-1}(N_F(z_{S,i}) \cap V(F'_{h-1}))\}.$$

For each $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $J \subseteq H$, let $J \oplus W_S = (H \oplus W_S)[V(J) \cup W_S]$. Let $\mathcal{H}^+ = \{H \oplus W_S : H \in \mathcal{H}, S \in \mathcal{S}\}$, and note that if any graph from $\mathcal{H}^+$ appears in G_h, then we can indeed extend our current embedding to one more dense spot in $\mathcal{S}$, and hence are done.

Let $\mathcal{J} = \{H \cap H' : H, H' \in \mathcal{H}, e(H \cap H') > 0\}$ and $\mathcal{J}' = \{H \cap H' : H, H' \in \mathcal{H}, H \neq H'\} \setminus \emptyset$. We will show that $\mathbb{P}(\exists H \in \mathcal{H}^+ \text{ with } H \subseteq G_h) = 1 - \exp(-\omega(n))$ follows from Lemma 2.2 and the following claim, which we then prove.

CLAIM 6.1.

(i) *For each* $J \in \mathcal{J}$, $2e(J) < (\Delta + 1)(v(J) - 1)$.

(ii) *For each* $H \in \mathcal{H}$ *and* $S \in \mathcal{S}$, $2e(H \oplus W_S) \leqslant (\Delta+1)s$.
(iii) *For each* $J \in \mathcal{J}'$ *and* $S \in \mathcal{S}$, $2e(J \oplus W_S) < (\Delta+1)v(J)$.

Note that, by (ii) of Claim 6.1, each graph in $\mathcal{H}^+$ has at most $(\Delta+1)s/2$ edges. We will now consider a subfamily $\mathcal{S}'$ of size at least $\frac{2|\mathcal{S}|}{(\Delta+1)s} = \Omega(n)$ of those copies of S from $\mathcal{S}$ so that $S \oplus W_S$ has the same number of edges, say m, where $1 \leqslant m \leqslant (\Delta+1)s/2$. Using (ii) of Claim 6.1, and that $q = \omega(n^{-\frac{2}{\Delta+1}})$, let the expected number of copies from $\mathcal{H}^+_{\mathcal{S}'} = \{H \oplus W_S : H \in \mathcal{H}, S \in \mathcal{S}'\}$ in G_h be denoted by μ, where

$$\mu = \sum_{S\in\mathcal{S}'} \sum_{H\in\mathcal{H}} q^{e(H\oplus W_S)} = \Omega(n^{s+1}q^m) = \Omega(n^{s+1}q^{(\Delta+1)s/2}) = \omega(n).$$

Let

$$\begin{aligned}
\delta &= \sum_{S,S'\in\mathcal{S}'} \sum_{\substack{H,H'\in\mathcal{H}\\ H\oplus W_S\sim H'\oplus W_{S'}}} q^{e(H\oplus W_S)+e(H'\oplus W_{S'})-e((H\oplus W_S)\cap(H'\oplus W_{S'}))} \\
&= q^{2m} \sum_{S,S'\in\mathcal{S}'} \sum_{\substack{H,H'\in\mathcal{H}\\ H\oplus W_S\sim H'\oplus W_{S'}}} q^{-e((H\oplus W_S)\cap(H'\oplus W_{S'}))} \\
&\leqslant q^{2m} \sum_{J\in\mathcal{J}} \sum_{\substack{S,S'\in\mathcal{S}'\\ S\neq S'}} \sum_{\substack{H,H'\in\mathcal{H}\\ H\cap H'=J}} q^{-e(J)} + q^{2m} \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} \sum_{\substack{H,H'\in\mathcal{H}\\ H\cap H'=J}} q^{-e(J\oplus W_S)} \\
&\leqslant q^{2m} \sum_{J\in\mathcal{J}} |\mathcal{S}'|^2 n^{2s-2v(J)} q^{-e(J)} + q^{2m} \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} n^{2s-2v(J)} q^{-e(J\oplus W_S)}.
\end{aligned} \tag{11}$$

Then, using (i) and (iii) of Claim 6.1, and as $\mu = \Omega(n^{s+1}q^m)$, we have

$$\begin{aligned}
\frac{\delta}{\mu^2} &= O\left(\sum_{J\in\mathcal{J}} |\mathcal{S}'|^2 n^{-2v(J)-2} q^{-e(J)} + \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} n^{-2v(J)-2} q^{-e(J\oplus W_S)}\right) \\
&= O\left(\sum_{J\in\mathcal{J}} n^{-2v(J)} q^{-(\Delta+1)(v(J)-1)/2} + \sum_{J\in\mathcal{J}'} |\mathcal{S}'| \cdot n^{-2v(J)-2} q^{-(\Delta+1)v(J)/2}\right) \\
&= o\left(\sum_{J\in\mathcal{J}} n^{-2v(J)} n^{v(J)-1} + \sum_{J\in\mathcal{J}'} n^{-2v(J)-1} n^{v(J)}\right) = o(n^{-1}).
\end{aligned}$$

Therefore, as $\mu = \omega(n)$ and $\frac{\delta}{\mu^2} = o(n^{-1})$, by Lemma 2.2, the probability that there is no graph in $\mathcal{H}^+_{\mathcal{S}'}$ in G_h is at most $\exp(-\frac{\mu^2}{4(\mu+\delta)}) = \exp(-\omega(n))$, as required. For Case I, it is left then only to prove Claim 6.1.

Proof of Claim 6.1. For (i), let $H \in \mathcal{H}$ be such that $J \subseteq H$. If $J \neq H$, and $v(J) \geqslant 3$, then, by (P2), we have $2e(J) \leqslant (\Delta+1)(v(J)-2) < (\Delta+1)(v(J)-1)$, as required. If $v(J) = 2$, then $(\Delta+1)(v(J)-1) = \Delta+1 > 2 \geqslant 2e(J)$.

Suppose then that $|J| = |H|$, so $v(J) = s$. If $s \leqslant \Delta$, then $2e(J) \leqslant s(s-1) < (s+1)(s-1) \leqslant (\Delta+1)(s-1)$, and if $s > \Delta+1$, then $2e(J) \leqslant s\Delta < s\Delta + s - (\Delta+1) = (\Delta+1)(s-1)$, as required. If $s = \Delta+1$, note that, as there is some edge between S_0 and F_{h-1} in F_h, we have that S_0, and hence J, is not a clique with $\Delta+1$ vertices. Thus, $2e(J) < s(s-1) = (\Delta+1)(s-1)$.

For (ii), suppose $s \geqslant \Delta + 1$. As H is dense, we have $2e(H) > (\Delta + 1)(s - 2)$, and thus

$$2e(H \oplus W_S) \leqslant 2\Delta s - 2e(H) < 2\Delta s - (\Delta + 1)(s - 2) = (\Delta + 1)s + 2(\Delta + 1 - s)$$
$$\leqslant (\Delta + 1)s,$$

as required.

So suppose $s \leqslant \Delta$. If $4 \leqslant s \leqslant \Delta - 1$, then, as $2e(H) > (\Delta + 1)(s - 2)$, we must have

$$s(s-1) > (\Delta + 1)(s - 2) \geqslant (s + 2)(s - 2) = s(s - 1) + s - 4 \geqslant s(s - 1),$$

a contradiction. If $s = 3$, then $2e(H) > \Delta + 1$ contradicts $\Delta \geqslant 5$.

Finally, if $s = \Delta$, then H must be the clique on Δ vertices because $2e(H) > (\Delta + 1)(\Delta - 2) = \Delta(\Delta - 1) - 2$. Therefore,

$$2e(H \oplus W_S) \leqslant 2\Delta^2 - 2e(H) = \Delta(\Delta + 1) = (\Delta + 1)s.$$

For (iii), let $H, H' \in \mathcal{H}$ be such that $H \cap H' = J$ and $H \neq H'$, which exist by the definition of $\mathcal{J}'$. Observe that $v(J) < s$ (since by our choice of $\mathcal{H}$ the vertex sets of any two copies are distinct). Let $I = H - V(J)$, and let $e(I, J)$ be the number of edges between I and J in H. Then,

$$2e(J \oplus W_S) \leqslant 2(\Delta v(J) - e(J) - e(J, I)) = 2(\Delta v(J) - e(H) + e(I))$$
$$= (\Delta + 1)v(J) + (\Delta - 1)v(J) - 2e(H) + 2e(I).$$

Thus, to prove the claim it is sufficient to show that $(\Delta - 1)v(J) < 2e(H) - 2e(I)$.

As H is dense, we have $2e(H) > (\Delta + 1)(v(J) + v(I) - 2)$. If $v(I) \geqslant 3$, then, from (P2), we have $2e(H) > (\Delta + 1)v(J) + 2e(I)$. If $v(I) = 2$, then $2e(H) > (\Delta + 1)v(J) \geqslant (\Delta - 1)v(J) + 2e(I)$.

Finally, suppose $v(I) = 1$, so that $e(I) = 0$ and $v(J) = s - 1$. By the reasoning in the proof of (ii), $s \geqslant \Delta$, otherwise we reach a contradiction. Thus, $2e(H) > (\Delta + 1)(s - 2) \geqslant (\Delta - 1)(s - 1) = (\Delta - 1)v(J) - 2e(I)$.

In each case, then, $2e(H) - 2e(I) > (\Delta - 1)v(J)$ as required. □

It remains to consider Case II. In this case, there is some graph from $\mathcal{S} \subseteq \mathcal{S}_h$ with no edges to F'_{h-1}. Therefore, it is sufficient for some graph in $\mathcal{H}$ to exist. Let $m = e(S_0)$ be the size of each (isomorphic) graph in $\mathcal{H}$, and note that $2m \leqslant \min\{s\Delta, s(s-1)\} \leqslant (s-1)(\Delta+1)$. Thus, we may take

$$\mu = \sum_{H \in \mathcal{H}} q^m = \Omega(n^s q^m) = \Omega(n^s q^{(s-1)(\Delta+1)/2}) = \omega(n).$$

Let $\mathcal{J} = \{H \cap H' : H, H' \in \mathcal{H}, e(H \cap H') > 0, H \neq H'\}$ and note that, if $J \in \mathcal{J}$ and $v(J) \geqslant 3$, then $2e(J) \leqslant (\Delta + 1)(v(J) - 2)$ by (P2). Let

$$\delta = \sum_{\substack{H,H' \in \mathcal{H} \\ H \sim H', H \neq H'}} q^{e(H)+e(H')-e(H \cap H')} = q^{2m} \sum_{J \in \mathcal{J}} \sum_{\substack{H,H' \in \mathcal{H} \\ H \cap H' = J}} q^{-e(J)} \leqslant q^{2m} \sum_{J \in \mathcal{J}} n^{2s-2v(J)} q^{-e(J)}$$

$$\leqslant q^{2m-1} n^{2s-2} + q^{2m} \sum_{J \in \mathcal{J} : v(J) \geqslant 3} n^{2s-2v(J)} q^{-(\Delta+1)(v(J)-2)/2}.$$

Then, as $\mu = \Omega(n^s q^m)$, we have

$$\frac{\delta}{\mu^2} = O\left(q^{-1}n^{-2} + \sum_{J\in\mathcal{J}:v(J)\geqslant 3} n^{-2v(J)}q^{-(\Delta+1)(v(J)-2)/2}\right) = o\left(n^{-1} + \sum_{J\in\mathcal{J}:v(J)\geqslant 3} n^{-v(J)-2}\right)$$
$$= o(n^{-1}).$$

Therefore, as $\mu = \omega(n)$, and $\frac{\delta}{\mu^2} = o(n^{-1})$, by Lemma 2.2, the probability that there is no graph in $\mathcal{H}$ in G_h is at most $\exp(-\frac{\mu^2}{2(\mu+\delta)}) = \exp(-\omega(n))$, as required. □

6.2. *Proof of Lemma* 5.4. Again, we use Janson's inequality and similar calculations to Ferber *et al.* [17].

Proof of Lemma 5.4. Recall that $W_{h-1} = [2n] \setminus g_{h-1}(F_{h-1})$. Let $1 \leqslant h \leqslant k, 1 \leqslant r \leqslant |\mathcal{S}'_h| \leqslant \frac{\varepsilon n}{s_h^2 k}$, $\mathcal{S} \subseteq \mathcal{S}'_h$ and $U \subseteq W_{h-1}$ with $|\mathcal{S}| = r$ and $|U| \leqslant s_h^2 r$. Note that, as $|U| \leqslant s_h^2 r \leqslant \varepsilon n$, we have $|U \cup (W_0 \setminus W_{h-1})| \leqslant 2\varepsilon n$. Therefore, by the property stated in (A2), for each $v \in V(F) \setminus V(F^*)$ and each $u \in [2n]$, we have

$$|N_G(u, B(v)) \cap (W_{h-1} \setminus U)| \geqslant 2\varepsilon n, \tag{12}$$

and, in particular, $|B(v) \cap (W_{h-1} \setminus U)| \geqslant 2\varepsilon n$ and we set $B'(v) = B(v) \cap (W_{h-1} \setminus U)$.

Let $s = s_h$. As in the proof of Lemma 5.3, we will consider two cases: Case I where each copy S from $\mathcal{S}$ has some edge between S and F_{h-1} in F_h and Case II when for some copy S from $\mathcal{S}$ there is no such edge.

Suppose first that we are in Case I. For all $S \in \mathcal{S}$, since $\Delta(F) \leqslant \Delta$ and $|S| = s$, there are certainly at most Δs vertices in F_{h-1} with some edge in S, and at most 2^s ways of attaching such a vertex to S. Thus, we can consider a subfamily $\mathcal{S}'$ of at least $\frac{1}{2^{\Delta s^2}}|\mathcal{S}| = \Omega(n)$ copies of S from $\mathcal{S}$ which are all isomorphic when the edges from S to F_h are added to S. Pick $S_0 \in \mathcal{S}'$. Label $V(S_0) = \{v_1, \dots, v_s\}$ so that v_1 has a neighbour in F_{h-1}. Recall that for $S \in \mathcal{S}$ we labelled $V(S) = \{z_{S,1}, \dots, z_{S,s_h}\}$. Without loss of generality, we can assume for each $S \in \mathcal{S}$ that $v_i \mapsto z_{S,i}$ is an isomorphism from S_0 into S, and that $z_{S,1}$ has a neighbour in F_h in $V(F_{h-1})$ (possible as we are in Case I). Let $\mathcal{H}$ be a set of $\binom{|U|}{s}$ copies of S_0 in the complete graph with vertex set U, where each copy of S_0 has a different vertex set. For each $H \in \mathcal{H}$, label $V(H) = \{v_{H,1}, \dots, v_{H,s}\}$ so that $v_i \mapsto v_{H,i}$ is an isomorphism of S_0 to H.

For each $S \in \mathcal{S}'$, pick the image w_S of an already embedded neighbour of the vertex $z_{S,1}$ corresponding to v_1, that is, pick $w_S \in g_{h-1}(N_{F_h}(z_{S,1}) \cap V(F_{h-1}))$. For each $S \in \mathcal{S}'$, let

$$\mathcal{H}_S = \{H \in \mathcal{H} : v_{H,1} \in N_G(w_S) \text{ and } v_{H,i} \in B'(v_{S,i}) \text{ for each } 1 \leqslant i \leqslant s\}.$$

For each $S \in \mathcal{S}'$, note that, from (12), we have $|\mathcal{H}_S| = \Omega(n^s)$.

For each $S \in \mathcal{S}'$, let $W_S = g_{h-1}(\bigcup_{v\in V(S)} N_{F_h}(v) \cap V(F_{h-1}))$ be the set of images of already embedded neighbours of vertices in S. For each $H \in \mathcal{H}_S$ and $S \in \mathcal{S}'$, let $H \oplus W_S$ be the graph with vertex set $V(H) \cup W_S$ and edge set

$$E(H) \cup (\{v_{H,i}v : 1 \leqslant i \leqslant s, v \in g_{h-1}(N_{F_h}(w_{S,i}) \cap V(F_{h-1}))\} \setminus \{v_{H,1}w_S\}).$$

These are exactly the edges we need in order to extend our embedding of F_{h-1} to contain S embedded into H, as $v_{H,1}w_S \in E(G)$. Let $\mathcal{H}^+ = \{H \oplus W_S : S \in \mathcal{S}', H \in \mathcal{H}_S\}$, and note that if any graph $H \oplus W_S \in \mathcal{H}^+$ appears in G'_h then, as $v_{H,1}w_S \in E(G)$, $V(H) \in E(L_S)$, and we are done.

Let $\mathcal{J} = \{H \cap H' : H, H' \in \mathcal{H}, e(H \cap H') > 0\}$ and $\mathcal{J}' = \{H \cap H' : H, H' \in \mathcal{H}, H \neq H'\} \setminus \emptyset$. Note that (i) and (iii) of Claim 6.1 hold here as well. For each $H \in \mathcal{H}$

and $S \in \mathcal{S}'$, $E(H \oplus W_S)$ does not include $v_{H,1}w_S$, and, therefore, in place of (ii), the following holds.

(ii') For each $S \in \mathcal{S}$ and $H \in \mathcal{H}_S$, $2e(H \oplus W_S) \leqslant (\Delta+1)s - 2$.

Note that, by our choice of $\mathcal{S}'$, each graph in $\mathcal{H}^+$ has the same number of edges, m say. Note that, as the property we are looking for is monotone, we may assume that $q^{-1/2} = \omega(\log n)$. Using (ii'), let

$$\mu = \sum_{S\in\mathcal{S}'} \sum_{H\in\mathcal{H}_S} q^m = \Omega(rn^s q^m) = \Omega(rn^s q^{(\Delta+1)s/2-1}) = \Omega(rq^{-1}) = \omega(r\log n).$$

We remark that this is the only place where we use that the edge $v_{H,1}w_S$ is not included in $H \oplus W_S$, since it is already present in G.

Defining δ as follows, and using similar deductions to those used to reach (11), we have

$$\begin{aligned}\delta &= \sum_{S,S'\in\mathcal{S}'} \sum_{\substack{H\in\mathcal{H}_S, H'\in\mathcal{H}_{S'} \\ H\oplus W_S \sim H'\oplus W_{S'}}} q^{e(H\oplus W_S)+e(H'\oplus W_{S'})-e((H\oplus W_S)\cap(H'\oplus W_{S'}))} \\ &\leqslant q^{2m}r^2 \sum_{J\in\mathcal{J}} n^{2s-2v(J)} q^{-e(J)} + q^{2m} \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} n^{2s-2v(J)} q^{-e(J\oplus W_S)}.\end{aligned}$$

Then, using (i) and (iii) of Claim 6.1, and that $\mu = \Omega(rn^s q^m)$, we have

$$\begin{aligned}\frac{\delta}{\mu^2} &= O\left(\sum_{J\in\mathcal{J}} n^{-2v(J)} q^{-e(J)} + r^{-2} \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} n^{-2v(J)} q^{-e(J\oplus W_S)}\right) \\ &= O\left(\sum_{J\in\mathcal{J}} n^{-2v(J)} q^{-((\Delta+1)(v(J)-1)-1)/2} + r^{-2} \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} n^{-2v(J)} q^{-((\Delta+1)v(J)-1)/2}\right) \\ &= o\left(q^{1/2} \sum_{J\in\mathcal{J}} n^{-2v(J)} n^{v(J)-1} + q^{1/2}r^{-2} \sum_{J\in\mathcal{J}'} \sum_{S\in\mathcal{S}'} n^{-2v(J)} n^{v(J)}\right) \\ &= o(q^{1/2}n^{-1} + q^{1/2}r^{-1}) = o(r^{-1}\log^{-1} n).\end{aligned}$$

Therefore, as $\mu = \omega(r\log n)$, and $\frac{\delta}{\mu^2} = o(r^{-1}\log^{-1} n)$, by Lemma 2.2, the probability that there is no graph in $\mathcal{H}^+$ in G'_h is at most $\exp(-\frac{\mu^2}{2(\mu+\delta)}) = \exp(-\omega(r\log n))$, completing the proof of Lemma 5.4 in Case I.

Let us assume now we are in Case II, with some S_0 with no edges between $S_0 \in \mathcal{S}'$ and F_{h-1} in F_h. Label $V(S_0) = \{v_1, \dots, v_s\}$. Let $\mathcal{H}$ be a maximal set of copies of S_0 in the complete graph with vertex set U, where each copy H of S_0 has a different vertex set, $\{v_{H,1}, \dots, v_{H,s}\}$ say, so that $v_i \mapsto v_{H,i}$ is an embedding of S_0, and $v_{H,i} \in B'(v_{S_0,i})$ for each $1 \leqslant i \leqslant s$.

Note that if we have some graph $H \in \mathcal{H}$ in G'_h, then we are done, as then $V(H) \in E(L_{S_0})$. From (12), we have $|\mathcal{H}| = \Omega(n^s)$, so, with very similar calculations to Case II in the proof of Lemma 5.3, we have that the probability that there exists no graph from $\mathcal{H}$ in G'_h is at most $\exp(-\omega(n)) \leqslant \exp(-\omega(r\log(n/r))$, as required. □

§7. *Concluding remarks.*

Extending Theorem 1.3 to smaller maximum degrees. Theorem 1.3 can be easily extended to $\Delta \leqslant 3$ using basically the same approach as in §5. The definition of the "dense spots," however, has to be slightly adapted to each case, but since it is straightforward, we omit the

details. There is no extension to $\Delta = 4$ of Theorem 1.2 due to the existence of one problematic dense spot: a triangle attached to the rest of the graph with two pendant edges at each vertex. This means that, using a similarly defined set of subgraphs $\mathcal{F}$ as in the proof of Theorem 1.3, we cannot show that one of these subgraphs appears whp in $G(n, \omega(n^{-2/5}))$ (i.e., we cannot prove (A1) in Definition 3.2), and this prevents our approach from extending to this case.

Using our method. Our main technical theorem, Theorem 3.3, provides a new general purpose tool for finding spanning structures F in randomly perturbed graphs $G_\alpha \cup G(n, p)$. To use Theorem 3.3, it is sufficient to show that F has a collection of subgraphs which is (α, p)-suitable. Our approach avoids the regularity lemma, which appears in many previous proofs for results concerning spanning structures in $G_\alpha \cup G(n, p)$ [**5**, **27**, **28**]. In particular, our approach provides simpler proofs for recent results concerning bounded degree spanning trees and factors, as we sketch in the following.

Spanning trees. Krivelevich *et al.* [**28**] showed that, for any $\alpha, \Delta > 0$, if $p = \omega(1/n)$ and T is an n-vertex tree with maximum degree at most Δ, then $G_\alpha \cup G(n, p)$ contains a copy of T whp. We can reprove this result using Theorem 3.3 as follows. Fixing $\alpha > 0$ and $\Delta > 0$, let $\varepsilon = \varepsilon(\alpha, \Delta)$ be as given in Definition 3.2. Let $p = \omega(1/n)$ and let T be a tree with n vertices and maximum degree at most Δ. Clearly T contains some subtree T' with just over $(1 - \varepsilon)n$ vertices, pick such a subtree and let $\mathcal{F} = \{T'\}$. By the work of Alon *et al.* [**4**], we know that $G(n, p/2)$ whp contains a copy of T', and therefore (A1) holds for $\mathcal{F}$. Furthermore, (A2) easily holds without even recourse to the random edges in $G(2n, p/6)$. The copy of T' can be extended by iteratively adding leaves. When we wish to add a leaf to a vertex w, say, to embed $v \in V(F) \setminus V(F')$, as $|B(v) \cap N_G(w)| \geqslant 4\varepsilon n$, there will be many vertices to choose from in $B(v) \cap N_G(w)$ which are not yet in the embedding. Thus, $\mathcal{F}$ is (α, p)-suitable and Theorem 3.3 applies.

Factors. Balogh *et al.* [**5**] showed that for every H, if $p = \omega(n^{-1/m_1(H)})$, then $G_\alpha \cup G(n, p)$ contains an H-factor whp. Again, we can use Theorem 3.3 to easily reprove this result. Indeed, let F be an H-factor and $\mathcal{F}$ be the set of subgraphs of F consisting of disjoint copies of H which cover at least $(1 - \varepsilon)n$ vertices. By Theorem 2.3, we have that (A1) holds for $\mathcal{F}$. Another simple application of Janson's inequality gives that (A2) holds as well.

Randomly perturbed hypergraphs. Recently generalisations of the model of randomly perturbed graphs to hypergraphs attracted much attention. Again, the union of a binomial random r-uniform hypergraph $G^{(r)}(n, p)$ and a deterministic r-uniform hypergraph G_α satisfying a certain minimum degree condition is considered. In the hypergraph setting, several different notions of minimum degree are possible.

The study of randomly perturbed hypergraphs was initiated by Krivelevich *et al.* [**27**] who considered hypergraphs G_α with *collective minimum degree* αn, that is, each $(r - 1)$-set of vertices of G_α is contained in at least αn edges. A *loose Hamilton cycle* in an r-uniform hypergraph on $n = (r - 1)k$ vertices for some integer k, is a labelling of its vertices by $0, \dots, n - 1$ such that $\{i, \dots, i + (r - 1)\}$ is an edge for each $i = (r - 1)j$ with $j \in \{0, \dots, k - 1\}$, where indices are taken modulo n. In other words, consecutive edges of a loose Hamilton cycle overlap in exactly one vertex. We remark that, for loose Hamilton cycles, a Dirac-type theorem is known [**24**]. Krivelevich *et al.* [**27**] proved that, for any G_α with collective minimum degree αn, the addition of random edges with edge probability $c(\alpha)n^{-r+1}$ (where $c(\alpha) > 0$ depends on α only) is sufficient to create whp perfect matchings

as well as loose Hamilton cycles. Comparing this to the threshold for matchings and loose Hamilton cycles in random hypergraphs, which is $n^{-r+1}\log n$ [14, 19, 23]), this again differs by a factor of $\log n$.

Different minimum degree conditions were considered by McDowell and Mycroft [30]. An r-uniform hypergraph G_α has *minimum ℓ-degree* at least $\alpha n^{r-\ell}$ if each ℓ-set of vertices of G_α is contained in at least $\alpha n^{r-\ell}$ edges. An *ℓ-overlapping* cycle is defined analogously to a loose Hamilton cycle, but with consecutive edges overlapping in exactly ℓ vertices. A *tight Hamilton cycle* in an r-uniform hypergraph is an $(r-1)$-overlapping Hamilton cycle. McDowell and Mycroft [30] showed that for ℓ-overlapping Hamilton cycles with $\ell \geqslant 3$ it is possible to save a polynomial factor n^ε on the edge probability in randomly perturbed r-uniform hypergraphs $G_\alpha \cup G^{(r)}(n,p)$ compared to $G^{(r)}(n,p)$ alone, under the assumption that G_α has minimum ℓ-degree at least αn^ℓ and minimum $(r-\ell)$-degree at least $\alpha n^{r-\ell}$. This result was extended by Bedenknecht *et al.* [6] to powers of tight Hamilton cycles, with the additional assumption of collective minimum degree at least αn with $\alpha > c_{r,\ell}$.

The weaker notion of minimum 1-degree was studied in the context of randomly perturbed hypergraphs by Han and Zhao [21]. It is not difficult to see that an r-uniform hypergraph with minimum collective degree at least αn has minimum 1-degree at least $\alpha\binom{n-1}{r-1}$. Hence, Han and Zhao [21] strengthen the results of Krivelevich,*et al.* by proving that adding $c(\alpha)n$ random edges to G_α, whp creates a perfect matching and a loose Hamilton cycle. Furthermore, adding $c(\alpha)n^{r-1}$ random edges to G_α gives rise to a tight Hamilton cycle. Both these results, as well as those from [27], use the regularity method.

The absorption technique we introduce in this paper can be extended to the randomly perturbed hypergraph model, and may allow some progress. In particular, we have confirmed that an easy extension of our method gives the appearance threshold for a perfect matching and a loose Hamilton cycle in this model, recovering the results of [21, 27].

We note that the third and fourth of the current authors [34] have extended the result of Riordan [36] to hypergraphs. Similar extensions of Theorems 1.2 and 1.3 however remain open and would be very interesting.

Universality. We believe that a universality result corresponding to our main theorem holds as well. That is, we believe that when $p = \omega(n^{-\frac{2}{\Delta+1}})$ the randomly perturbed graph $G_\alpha \cup G(n,p)$ contains whp a copy of every graph in $\mathcal{F}(n,\Delta)$ simultaneously. However, our use of Riordan's result [36], which was proved by second moment calculations, makes it unlikely that our techniques can be used to obtain such a result. Thus, new ideas are required. Similarly, p_Δ is commonly believed to be the threshold for $G(n,p)$ to contain a copy of every graph in $\mathcal{F}(n,\Delta)$ simultaneously, but the current methods to attack this problem (see the discussion after Theorem 1.2) require an edge probability in distinct excess of this conjectured threshold.

In the case of spanning bounded degree trees, in joint work with Han and Kohayakawa, we establish the following universality result in [11]. We show that $G_\alpha \cup G(n, c(\alpha,\Delta)/n)$ simultaneously contains all spanning trees of maximum degree at most Δ.

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