JGSP **25** (2012) 1–18

EIGENVECTORS OF THE SO(3,$\mathbb{R}$) MATRICES

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Communicated by XXX

Abstract. Let $R = (r_{ij}) \in$ SO(3,$\mathbb{R}$). We give several different proofs of the fact that the vector

$$V := \Big(\frac{1}{r_{23}+r_{32}}, \frac{1}{r_{13}+r_{31}}, \frac{1}{r_{12}+r_{21}}\Big)^t$$

if it exists, is an eigenvector of R corresponding to the eigenvalue one.

MSC:15A18, 15-01, 97Axx

Keywords: Eigenvectors, orthogonal matrices, rotations in $\mathbb{R}^3$

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1. Introduction

Let R be a 3×3 real matrix and suppose that we want to find an eigenvector V for R. Every student learns an algorithm for this, but is it possible to skip the toil, and write down V explicitly in terms of r_{ij}? For example, we can easily do this for a matrix of rank 1. If X is a nonzero column, then we can simply take $V = X$. Indeed, we know that $R = XY^t$ for some vector Y and

$$RX = XY^tX = X\langle Y, X\rangle = \langle Y, X\rangle X$$

where we have used that the 1×1 matrix Y^tX can be identified with the inner product $\langle Y, X\rangle$. Another interesting example is when we consider skew-symmetric matrices:

Theorem 1 *For any* 3×3 *skew-symmetrical matrix*

$$Q = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$$

the vector

$$V = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

belongs to its kernel, thus $QV = 0$.

This can be checked directly, but in fact we can generalise this to any matrix of rank 2.

Theorem 2 *Let* $R_{ij} = (-1)^{i+j}D_{ij}$ *where* D_{ij} *is a minor obtained by deleting the row* i *and column* j *from the matrix* R. *If* R *has* rank 2, *then all three vectors* $V_j = (R_{j1},\ R_{j2},\ R_{j3})^t$ *belong to its kernel and at least one of them is non-zero eigenvector.*

Proof: It is well-known that (see for example [6, Theorem 3.15, p.69])

$$\sum_{k=1}^{3} r_{ik}R_{jk} = \delta_{ij}\det R$$

where δ_{ij} is 1 if $i = j$ and 0 otherwise. In the case of rank 2, we get that $\det\ R = 0$ implies $RV_j = 0$, and at least one of the vectors V_j is non-zero. ■

What can be said about non-singular matrices? If we know an eigenvalue λ we can simply apply the same arguments to the matrix $R - \lambda I$ to find the eigenvector (the case $R = \lambda I$ will be special, but here we can take any non-zero vector). We always know an eigenvalue ± 1 for an orthogonal matrices. For example it is well-known that $R \in$SO(3,$\mathbb{R}$) describes a rotation in $\mathbb{R}^3$ about some axis described by a vector V (see e.g. [1, Thm. 5.5, p.124]), and this V is an eigenvector of A corresponding to the eigenvalue 1. So we want to express axis of rotation in terms of the matrix entries of R. But unexpectedly, we can get the vector V quite easily.

Theorem 3 *Let R=$(r_{ij}) \in$ SO(3,$\mathbb{R}$). Let*

$$\begin{aligned} V &= \left(\frac{1}{r_{23}+r_{32}},\, \frac{1}{r_{13}+r_{31}},\, \frac{1}{r_{12}+r_{21}} \right)^t \\ U &= \left(r_{23}-r_{32},\, r_{31}-r_{13},\, r_{12}-r_{21} \right)^t \\ W_1 &= \left(1+r_{11}-r_{22}-r_{33},\, r_{12}+r_{21},\, r_{13}+r_{31} \right)^t \\ W_2 &= \left(r_{12}+r_{21},\, 1+r_{22}-r_{11}-r_{33},\, r_{23}+r_{32} \right)^t \\ W_3 &= \left(r_{13}+r_{31},\, r_{23}+r_{32},\, 1+r_{33}-r_{11}-r_{22} \right)^t. \end{aligned}$$

Then $RV = V, RU = U, RW_i = W_i$, so any of these vectors (if it exists and is non-zero), is an eigenvector with eigenvalue one. If $R \neq I$ then at least one of them exists and is non-zero.

The most unexpected one is the vector V so we concentrate on it.

Theorem 4 *Let $R = (r_{ij}) \in$ SO(3,$\mathbb{R}$). If the vector*

$$V = \Big(\frac{1}{r_{23}+r_{32}}, \frac{1}{r_{13}+r_{31}}, \frac{1}{r_{12}+r_{21}}\Big)^t$$

exists (that is, the denominators are non-zeros), then $RV = V$.

In fact, this result appears as an exercise in M. Artin's classic textbook *Algebra* [1, Ex.14, §5, Chap.4, p.149]. Our plan is to give several different proofs of Theorem 4 obtaining simultaneously the proof of Theorem 3.

2. Two Algebraic Proofs

We start from some useful statements.

Theorem 5 *For arbitrary n and any $R = (r_{ij}) \in \mathrm{SO}(n,\mathbb{R})$, one has $R_{ij} = r_{ij}$, where $R_{ij} = (-1)^{i+j} D_{ij}$ and D_{ij} is a minor obtained by deleting the row i and column j from the matrix R.*

Proof: It is well-known that for any invertible matrix, $R^{-1} = \frac{1}{\det R}(R_{ij})^t$. In our case $\det R = 1$ and $R^{-1} = R^t$, which proves the claim. ■

Lemma 6 *Let $R = (r_{ij}) \in \mathrm{SO}(n,\mathbb{R})$. Let i, j, k be three different indices between 1 and 3. Then*

$$\begin{aligned}
(1 + r_{ii})(r_{jk} + r_{kj}) &= r_{ij}r_{ki} + r_{ji}r_{ik} \\
(r_{jj} + r_{kk})(r_{jk} + r_{kj}) &= -(r_{ij}r_{ik} + r_{ji}r_{ki}) \\
(r_{ij}^2 + r_{ik}^2)(r_{ij}r_{ik} + r_{ji}r_{ki}) &= (r_{ij}r_{ki} + r_{ji}r_{ik})(r_{ij}r_{ji} + r_{ik}r_{ki}).
\end{aligned}$$

Proof: By symmetry, it is sufficient to consider the case $i = 1, j = 2, k = 3$ only. Using the previous theorem we have

$$\begin{aligned}
r_{23} + r_{32} = R_{23} + R_{32} &= -(r_{11}r_{32} - r_{12}r_{31}) - (r_{11}r_{23} - r_{21}r_{13}) \\
&= -r_{11}(r_{23} + r_{32}) + r_{12}r_{31} + r_{21}r_{13}.
\end{aligned}$$

Consequently, $(1 + r_{11})(r_{23} + r_{32}) = r_{12}r_{31} + r_{21}r_{13}$.
The second equality follows from the orthogonality

$$(r_{22}+r_{33})(r_{23}+r_{32}) = (r_{22}r_{23}+r_{33}r_{33}) + (r_{22}r_{32}+r_{23}r_{33}) = -r_{21}r_{31} - r_{12}r_{13}$$

and we are done.
For the last equality we write

$$\begin{aligned}
& (r_{12}^2 + r_{13}^2)(r_{13}r_{12} + r_{31}r_{21}) \\
& = (r_{12}r_{31} + r_{21}r_{13})(r_{12}r_{21} + r_{13}r_{31}) \\
\Leftrightarrow \quad & r_{12}^3 r_{13} + r_{12}^2 r_{31}r_{21} + r_{13}^3 r_{12} + r_{13}^2 r_{31}r_{21} \\
& = r_{12}^2 r_{31}r_{21} + r_{21}^2 r_{12}r_{13} + r_{31}^2 r_{13}r_{12} + r_{13}^2 r_{31}r_{21} \\
\Leftrightarrow \quad & r_{12}^3 r_{13} + r_{13}^3 r_{12} = r_{21}^2 r_{12}r_{13} + r_{31}^2 r_{13}r_{12} \\
\Leftrightarrow \quad & r_{12}r_{13}(r_{12}^2 + r_{13}^2) = r_{12}r_{13}(r_{21}^2 + r_{31}^2) \\
\Leftrightarrow \quad & r_{12}r_{13}(1 - r_{11}^2) = r_{12}r_{13}(1 - r_{11}^2)
\end{aligned}$$

where we used the orthogonality conditions. ■

Now we are ready for the **first proof** of Theorem 4.

Proof: We have

$$RV = \begin{pmatrix} \dfrac{r_{11}}{r_{23}+r_{32}} + \dfrac{r_{12}}{r_{13}+r_{31}} + \dfrac{r_{13}}{r_{12}+r_{21}} \\ \dfrac{r_{21}}{r_{23}+r_{32}} + \dfrac{r_{22}}{r_{13}+r_{31}} + \dfrac{r_{23}}{r_{12}+r_{21}} \\ \dfrac{r_{31}}{r_{23}+r_{32}} + \dfrac{r_{32}}{r_{13}+r_{31}} + \dfrac{r_{33}}{r_{12}+r_{21}} \end{pmatrix}.$$

We want to prove that

$$\frac{r_{11}}{r_{23}+r_{32}} + \frac{r_{12}}{r_{13}+r_{31}} + \frac{r_{13}}{r_{12}+r_{21}} = \frac{1}{r_{23}+r_{32}}$$

(the proofs for other coordinates are similar). Suppose first that $r_{11}+1 \neq 0$. Then this is equivalent to

$$\frac{(1-r_{11})(1+r_{11})}{(1+r_{11})(r_{23}+r_{32})} = \frac{r_{12}}{r_{13}+r_{31}} + \frac{r_{13}}{r_{12}+r_{21}}.$$

By Lemma 6 this transforms to

$$\begin{aligned} \frac{1-r_{11}^2}{r_{12}r_{31}+r_{21}r_{13}} &= \frac{r_{12}^2+r_{13}^2+r_{12}r_{21}+r_{13}r_{31}}{(r_{13}+r_{31})(r_{12}+r_{21})} \\ \Leftrightarrow (r_{12}^2+r_{13}^2)\left(\frac{1}{r_{12}r_{31}+r_{21}r_{13}} - \frac{1}{(r_{13}+r_{31})(r_{12}+r_{21})}\right) &= \frac{r_{12}r_{21}+r_{13}r_{31}}{(r_{13}+r_{31})(r_{12}+r_{21})} \\ \Leftrightarrow \qquad \frac{(r_{12}^2+r_{13}^2)(r_{13}r_{12}+r_{31}r_{21})}{(r_{12}r_{31}+r_{21}r_{13})(r_{13}+r_{31})(r_{12}+r_{21})} &= \frac{r_{12}r_{21}+r_{13}r_{31}}{(r_{13}+r_{31})(r_{12}+r_{21})}. \end{aligned}$$

This is equivalent to

$$(r_{12}^2+r_{13}^2)(r_{13}r_{12}+r_{31}r_{21}) = (r_{12}r_{31}+r_{21}r_{13})(r_{12}r_{21}+r_{13}r_{31})$$

and we can apply Lemma 6 again.

It remains to consider the case $r_{11} = -1$. But then

$$r_{12}^2 + r_{13}^2 = 1 - r_{11}^2 = 0 \Rightarrow r_{12} = r_{13} = 0.$$

Similarly we get $r_{21} = r_{31} = 0$. But this contradicts $r_{12}+r_{21} \neq 0$. ■

So straightforward calculations was not so obvious as expected. We can slightly improve them in our **second proof**.

Proof: If we apply Theorem 2 to the matrix $R - I$ which has rank 2 we get the eigenvector directly. Suppose that this is for example

$$\begin{aligned} V_1 &= \left(\begin{vmatrix} r_{22}-1 & r_{23} \\ r_{32} & r_{33}-1 \end{vmatrix}, -\begin{vmatrix} r_{21} & r_{23} \\ r_{31} & r_{33}-1 \end{vmatrix}, \begin{vmatrix} r_{21} & r_{22}-1 \\ r_{31} & r_{32} \end{vmatrix} \right)^t \\ &= (R_{11}+1-r_{22}-r_{33},\ R_{12}+r_{21},\ R_{13}+r_{31})^t \\ &= (1+r_{11}-r_{22}-r_{33},\ r_{12}+r_{21},\ r_{13}+r_{31})^t \end{aligned}$$

obtaining the vector W_1 from Theorem 3, so we get part of this theorem as well. Vectors V_2, V_3 lead us naturally to W_2, W_3. To finish the proof of Theorem 4, we divide the obtained vector by $(r_{12} + r_{21})(r_{13} + r_{31})$ (which is non-zero), and it remains to show that

$$\frac{1 + r_{11} - r_{22} - r_{33}}{(r_{12} + r_{21})(r_{13} + r_{31})} = \frac{1}{r_{23} + r_{32}}.$$

By Lemma 6 we have

$$\begin{aligned}
(1 + r_{11} - r_{22} - r_{33})&(r_{23} + r_{32}) \\
&= (1 + r_{11})(r_{23} + r_{32}) - (r_{22} + r_{33})(r_{23} + r_{32}) \\
&= r_{12}r_{31} + r_{21}r_{13} + r_{21}r_{31} + r_{12}r_{13} \\
&= (r_{12} + r_{21})(r_{13} + r_{31})
\end{aligned}$$

which finishes the proof. ■

3. Origin of the Non-trivial Eigenvector

Now we want to understand the origin of this non-trivial eigenvector. We find one possible source in skew-symmetric matrices.

Theorem 7 *Let R be an orthogonal matrix (of any arbitrary size). If the vector $U \in \ker(R - R^t)$, then $R^2U = U$. Moreover, if R has only one real eigenvalue λ, then $RU = \lambda U$.*

Proof: We have

$$(R - R^t)U = 0 \Leftrightarrow RU = R^tU \Leftrightarrow R^2U = U$$

which proves the first statement.

Let $\{e_i\}$ be a (complex) basis of eigenvectors (which exists because R is a normal matrix). If $U = \sum x_i e_i$, then

$$R^2U - U = \sum x_i(\lambda_i^2 - 1)e_i = 0$$

which means that all x_i corresponding to complex eigenvalues λ_i should be equal to zero and U is proportional to the only eigenvector with real eigenvalue. ■

Now we are ready for the **third proof** of Theorem 4.

Proof: Suppose first that $R \neq R^t$, that is, $R^2 \neq I$. Then R has some complex eigenvalue λ. It follows that $\overline{\lambda}$ is another eigenvalue, and the third one is 1 (because $|\lambda| = 1$ and $\det R = 1$). Since

$$U = \begin{pmatrix} r_{23} - r_{32} \\ r_{31} - r_{13} \\ r_{12} - r_{21} \end{pmatrix} \in \ker(R - R^t)$$

by Theorem 1, and is a non-zero vector, we can apply Theorem 7 to get $RU = U$. We need only to show that $cV = U$ for some non-zero c. We put $c = r_{23}^2 - r_{32}^2$, and note that $c = r_{31}^2 - r_{13}^2$, $c = r_{12}^2 - r_{21}^2$ as well, for example

$$r_{23}^2 - r_{32}^2 = r_{31}^2 - r_{13}^2 \Leftrightarrow r_{13}^2 + r_{23}^2 = r_{31}^2 + r_{32}^2 \Leftrightarrow 1 - r_{33}^2 = 1 - r_{33}^2.$$

Then

$$\begin{aligned} cV &= \left(\frac{c}{r_{23} + r_{32}}, \frac{c}{r_{13} + r_{31}}, \frac{c}{r_{12} + r_{21}} \right)^t \\ &= \left(\frac{r_{23}^2 - r_{32}^2}{r_{23} + r_{32}}, \frac{r_{31}^2 - r_{13}^2}{r_{13} + r_{31}}, \frac{r_{12}^2 - r_{21}^2}{r_{12} + r_{21}} \right)^t = U. \end{aligned}$$

It remains to consider the case $R = R^t$, that is, $r_{ij} = r_{ji}$, and we need to prove that for

$$V' = \left(\frac{1}{r_{23}}, \frac{1}{r_{13}}, \frac{1}{r_{12}} \right)^t$$

we have $RV' = V'$. This can be done explicitly, for example for the first coordinate we have

$$\frac{r_{11}}{r_{23}} + \frac{r_{12}}{r_{13}} + \frac{r_{13}}{r_{12}} = \frac{1}{r_{23}} \Leftrightarrow \frac{r_{12}^2 + r_{13}^2}{r_{12}r_{13}} = \frac{1 - r_{11}}{r_{23}} \Leftrightarrow \frac{1 - r_{11}^2}{r_{12}r_{13}} = \frac{1 - r_{11}}{r_{23}}$$

So we need only to prove

$$(1 + r_{11})r_{23} = r_{12}r_{13} \Leftrightarrow r_{23} = r_{12}r_{31} - r_{11}r_{32} \Leftrightarrow r_{23} = R_{23}$$

which follows from Theorem 5. Note also that we have completed also the proof of Theorem 3 regarding the vector U. ■

4. A Geometric Interpretation of the Eigenvector

Now we want to find some geometrical interpretation of our eigenvector and consider **fourth proof** of Theorem 4.

Proof: The starting point is that any matrix $R \in$ SO(3,$\mathbb{R}$) can be written as a product of two reflections. (This is easy to see in the plane, and as every rotation in $\mathbb{R}^3$ has an axis of rotation, the result for rotations in $\mathbb{R}^3$ follows from the planar case.) So let X, Y be two unit vectors such that $R = (I - 2XX^t)(I - 2YY^t)$. The case when X and Y are proportional is not interesting for us (in this case $R = I$). So we suppose that they are linear independent and let $Z = X \times Y$ be their (nonzero) vector product. First we note that Z is the eigenvector we are looking for. Indeed, $X^t Z = \langle X, Z\rangle = 0$ and similarly $Y^t Z = 0$, giving, for some B, C, that

$$RZ = (I + BX^t + CY^t)Z = IZ = Z.$$

As we know that $Z = \begin{pmatrix} x_2y_3 - x_3y_2, \; x_3y_1 - x_1y_3, \; x_1y_2 - x_2y_1 \end{pmatrix}^t$ we need only to prove that our vector v is proportional to this one, that is,

$$\det \begin{pmatrix} v_i & z_i \\ v_j & z_j \end{pmatrix} = 0.$$

By symmetry, it is sufficient to consider the case $i = 1, j = 2$ only. We have

$$\det \begin{pmatrix} \dfrac{1}{r_{23} + r_{32}} & x_2y_3 - x_3y_2 \\ \dfrac{1}{r_{13} + r_{31}} & x_3y_1 - x_1y_3 \end{pmatrix} = 0$$

$$\Leftrightarrow (x_3y_1 - x_1y_3)(r_{13} + r_{31}) = (x_2y_3 - x_3y_2)(r_{23} + r_{32}).$$

Let $c = \langle X, Y\rangle$. Then $R = I - 2XX^t - 2YY^t + 4cXY^t$ and for $i \neq j$

$$r_{ij} + r_{ji} = -4x_ix_j - 4y_iy_j + 4c(x_iy_j + x_jy_i).$$

Our aim is

$$\begin{aligned}
&(x_3y_1 - x_1y_3)(-x_1x_3 - y_1y_3) + c(x_3y_1 + x_1y_3)\\
&\qquad = (x_2y_3 - x_3y_2)(-x_2x_3 - y_2y_3) + c(x_2y_3 + x_3y_2)\\
\Leftrightarrow\ & x_1y_1(-x_3^2 + y_3^2) + x_3y_3(-y_1^2 + x_1^2) + c((x_3y_1)^2 - (x_1y_3)^2)\\
&\qquad = x_3y_3(-x_2^2 + y_2^2) + x_2y_2(-y_3^2 + x_3^2) + c((x_2y_3)^2 - (x_3y_2)^2)\\
\Leftrightarrow\ & (x_1y_1 + x_2y_2)(-x_3^2 + y_3^2) + x_3y_3(-y_1^2 + x_1^2 + x_2^2 - y_2^2)\\
&\qquad = c(y_3^2(x_1^2 + x_2^2)) - x_3^2(y_1^2 + y_2^2)).
\end{aligned}$$

Now we use the fact that we have unit vectors. We have

$$\begin{aligned}
&(x_1y_1 + x_2y_2)(-x_3^2 + y_3^2) \; + \; x_3y_3(1 + y_3^2 - 1 - x_3^2)\\
&\qquad\qquad = \; c(y_3^2(1 - x_3^2)) - x_3^2(1 - y_3^2))\\
\Leftrightarrow \quad & (x_1y_1 + x_2y_2 + x_3y_3)(-x_3^2 + y_3^2) \; = \; c(y_3^2 - x_3^2)
\end{aligned}$$

and we are done because $c = x_1y_1 + x_2y_2 + x_3y_3$. ∎

5. A Proof Using the Lie Algebra of the Rotation Group

Define the Lie algebra

$$\mathfrak{so}(3,\mathbb{R}) := \{Q \in \mathbb{R}^{3\times 3};\, Q + Q^t = 0\}$$

of the Lie group SO(3,$\mathbb{R}$). We recall the following well-known result (see for example [8, Lemma 1B,p.31]).

Proposition 8 *Let $R = (r_{ij}) \in$ SO(3,$\mathbb{R}$). Then there exists a $t \in [0, 2\pi)$ and there exists a $Q \in \mathfrak{so}(3,\mathbb{R})$ such that $R = \mathrm{e}^{tQ}$. Moreover, if Q has the form*

$$Q = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$$

then taking the vector U as $U := (p \quad q \quad r)^t \in \mathbb{R}^3$, we have that R is a rotation about U through the angle t using the right-hand rule.

We will also need the fact that for $t \geq 0$

$$\mathrm{e}^{tQ} = \mathcal{L}^{-1}((sI - Q)^{-1})(t)$$

where $\mathcal{L}^{-1}$ denotes the (entrywise) inverse one-sided Laplace transform. The following fact is well-known (see for example, [2, §27,p.218]).

Proposition 9 *For large enough s,* $\displaystyle\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{tQ}\mathrm{d}t = (sI - Q)^{-1}$.

In the above, the integral of a matrix whose elements are functions of t is defined entrywise. If s is not an eigenvalue of Q, then $sI - Q$ is invertible, and by Cramer's rule

$$(sI - Q)^{-1} = \frac{1}{\det(sI - Q)}\mathrm{adj}(sI - Q).$$

So we see that each entry of $\mathrm{adj}(sI - Q)$ is a polynomial in s whose degree is at most $n - 1$, where n denotes the size of Q, that is, Q is an $n \times n$ matrix. Consequently, each entry m_{ij} of $(sI - Q)^{-1}$ is a rational function in s, whose inverse Laplace transform gives the matrix exponential e^{tQ}. We now give the **fifth proof** of Theorem 4.

Proof: Let Q, U be as in Proposition 8. By Cramer's rule

$$\begin{aligned}(sI-Q)^{-1} &= \begin{pmatrix} s & r & -q \\ -r & s & p \\ q & -p & s \end{pmatrix}^{-1} \\ &= \frac{1}{\det(sI-Q)} \begin{pmatrix} s^2+p^2 & rs+pq & -qs+rp \\ -rs+pq & s^2+q^2 & ps+qr \\ qs+rp & -ps+qr & s^2+r^2 \end{pmatrix}.\end{aligned}$$

Hence

$$R = \mathrm{e}^{tQ} = \mathcal{L}^{-1}\left(\frac{1}{\det(sI-Q)} \begin{pmatrix} s^2+p^2 & rs+pq & -qs+rp \\ -rs+pq & s^2+q^2 & ps+qr \\ qs+rp & -ps+qr & s^2+r^2 \end{pmatrix}\right)(t).$$

This yields

$$V = \begin{pmatrix} \dfrac{1}{r_{23}+r_{32}} \\ \dfrac{1}{r_{31}+r_{13}} \\ \dfrac{1}{r_{12}+r_{21}} \end{pmatrix} = \underbrace{\left(\mathcal{L}^{-1}\left(\frac{1}{\det(sI-Q)}\right)(t)\right)^{-1}}_{=:c} \begin{pmatrix} \dfrac{1}{2qr} \\ \dfrac{1}{2rp} \\ \dfrac{1}{2pq} \end{pmatrix} = \frac{c}{2pqr}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

which is a multiple of U. ■

6. A Quaternionic Proof

Let $\mathbf{D} := \{\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}; a, b, c, d \in \mathbb{R}\}$ be the ring of all quaternions, with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{i}\cdot\mathbf{j} = -\mathbf{j}\cdot\mathbf{i} = \mathbf{k}$, $\mathbf{j}\cdot\mathbf{k} = -\mathbf{k}\cdot\mathbf{j} = \mathbf{i}$, $\mathbf{k}\cdot\mathbf{i} = -\mathbf{i}\cdot\mathbf{k} = \mathbf{j}$. We define the norm of $\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ by

$$|\mathbf{q}| = \sqrt{a^2+b^2+c^2+d^2}$$

and the conjugate $\overline{\mathbf{q}}$ of $\mathbf{q}$ by

$$\overline{\mathbf{q}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

It can be checked that for $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{D}$, $|\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1||\mathbf{q}_2|$ and $|\mathbf{q}|^2 = \mathbf{q}\overline{\mathbf{q}}$. We identify $\mathbb{R}^3$ as a subset of $\mathbf{D}$ via

$$\mathbb{R}^3 = \{b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{D}; b, c, d \in \mathbb{R}\}.$$

If $|\mathbf{q}| = 1$ then for any $\mathbf{w} \in \mathbb{R}^3$, $\mathbf{q}\mathbf{w}\mathbf{q}^{-1} \in \mathbb{R}^3$, for example

$$\begin{aligned}\mathbf{q}\mathbf{i}\mathbf{q}^{-1} = \mathbf{q}\mathbf{i}\overline{\mathbf{q}} &= (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})\mathbf{i}(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}) \\ &= (a\mathbf{i} - b - c\mathbf{k} + d\mathbf{j})(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}) \\ &= a^2\mathbf{i} + ab - ac\mathbf{k} + ad\mathbf{j} - ba + b^2\mathbf{i} + bc\mathbf{j} + bd\mathbf{k} \\ &\quad -ca\mathbf{k} + cb\mathbf{j} - c^2\mathbf{i} - cd + da\mathbf{j} + db\mathbf{k} + dc - d^2\mathbf{i} \\ &= (a^2 + b^2 - c^2 - d^2)\mathbf{i} + 2(ad + bc)\mathbf{j} + 2(bd - ac) \in \mathbb{R}^3.\end{aligned}$$

So the map $T_{\mathbf{q}} : \mathbf{w} \mapsto \mathbf{q}\mathbf{w}\mathbf{q}^{-1}$ maps vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$ and clearly is linear. In fact, this collection of maps $T_{\mathbf{q}}$, $|\mathbf{q}| = 1$, is precisely the set SO(3,$\mathbb{R}$) of rotations in $\mathbb{R}^3$!

To see this note first that if $\mathbf{w} \in \mathbb{R}^3$, then its Euclidean norm $\|\mathbf{w}\|_2$ coincides with its quaternionic norm. Therefore $T_{\mathbf{q}}$ is also a rigid motion, since

$$\|T_{\mathbf{q}}\mathbf{w}\|_2 = |T_{\mathbf{q}}\mathbf{w}| = |\mathbf{q}\mathbf{w}\mathbf{q}^{-1}| = |\mathbf{q}||\mathbf{w}||\mathbf{q}^{-1}| = |\mathbf{w}| = \|\mathbf{w}\|_2$$

so our map corresponds to an orthogonal matrix. But because

$$T_{\mathbf{q}}(\mathbf{q} - a) = \mathbf{q}(\mathbf{q} - a)\mathbf{q}^{-1} = \mathbf{q}^2\mathbf{q}^{-1} - a\mathbf{q}\mathbf{q}^1 = \mathbf{q} - a$$

we have an invariant vector as well (when $\mathbf{q} = a$ we can take any vector), so our matrix belongs to SO(3,$\mathbb{R}$) and is a rotation. We can describe it explicitly.

Since $|a| \leq 1$, we can find a unique $t \in [0, 2\pi)$ such that $\cos\dfrac{t}{2} = a$ to get

$$\mathbf{q} = \left(\cos\frac{t}{2}\right) + \mathbf{v}.$$

We leave to the reader to prove that the angle of rotation around $\mathbf{v}$ is exactly t. It is clear that every rotation then arises in this manner.

Now we are ready to give the **sixth proof** of Theorem 4.

Proof: We need to consider the case $\mathbf{v} \neq 0$ only. By feeding in $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into $T_{\mathbf{q}}$, we can now compute the matrix R of $T_{\mathbf{q}}$ in terms of the entries of $\begin{pmatrix} b, & c, & d \end{pmatrix}^t$, where $\mathbf{v} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. We already know the first column and the rest we get by cyclic symmetry, and so

$$R = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(ac + bd) \\ 2(bc + ad) & a^2 + c^2 - b^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(ab + cd) & a^2 + d^2 - b^2 - c^2 \end{pmatrix}.$$

Now it is easy to check that

$$V = \begin{pmatrix} \dfrac{1}{r_{23}+r_{32}} \\ \dfrac{1}{r_{31}+r_{13}} \\ \dfrac{1}{r_{12}+r_{21}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{4cd} \\ \dfrac{1}{4bd} \\ \dfrac{1}{4bc} \end{pmatrix} = \frac{1}{4bcd}\begin{pmatrix} b \\ c \\ d \end{pmatrix}$$

which is a multiple of $\mathbf{v}$. ■

7. A Proof Using the Cayley Transform

We only consider the case when -1 is not eigenvalue of R, since the case when -1 is an eigenvalue of R (implying that $R^2 = I$) has been covered before in our third proof.

Theorem 10 *If $R = (r_{ij}) \in$ SO(3,$\mathbb{R}$) such that -1 is not an eigenvalue of R, then there exists a skew-symmetric Q such that $R = (I+Q)(I-Q)^{-1}$.*

Proof: As -1 is not an eigenvalue of R, $R + I$ is invertible. Define

$$Q = (R - I)(R + I)^{-1}.$$

Then

$$\begin{aligned} Q + Q^t &= (R-I)(R+I)^{-1} + (R^t+I)^{-1}(R^t-I) \\ &= (R-I)(R+I)^{-1} + (R^{-1}+I)^{-1}(R^{-1}-I) \\ &= (R-I)(R+I)^{-1} + (I+R)^{-1}RR^{-1}(I-R) \\ &= (R-I)(R+I)^{-1} + (I+R)^{-1}(I-R) = 0 \end{aligned}$$

where we use the commutativity to get the last equality. So Q is skew-symmetric. But then $I - Q$ is invertible. From the definition of Q, $Q(R + I) = R - I$, and solving for R, we obtain $R = (I + Q)(I - Q)^{-1}$. ■

Now we are ready to give the **seventh proof** of Theorem 4.

Proof: Given R, we can write A as $R = (I + Q)(I - Q)^{-1}$ for some skew-symmetric Q

$$Q = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} R =& (I+Q)(I-Q)^{-1} \\ =& \frac{1}{1+p^2+q^2+r^2}\begin{pmatrix} 1+p^2-q^2-r^2 & 2pq-2r & 2rp+2q \\ 2pq+2r & 1-p^2+q^2-r^2 & 2qr-2p \\ 2rp-2q & 2qr+2p & 1-p^2-q^2+r^2 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} \dfrac{1}{r_{23}+r_{32}} \\ \dfrac{1}{r_{31}+r_{13}} \\ \dfrac{1}{r_{12}+r_{21}} \end{pmatrix} = (1+p^2+q^2+r^2)\begin{pmatrix} \dfrac{1}{4qr} \\ \dfrac{1}{4rp} \\ \dfrac{1}{4pq} \end{pmatrix} = \frac{1+p^2+q^2+r^2}{4pqr}\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

which is an eigenvector of R corresponding to eigenvalue one, by Theorem 1. ■

8. A Proof Using Contour Integral of the Resolvent

We recall the following; see for example [5, §8.2, p.127].

Proposition 11 *For an isolated eigenvalue of a square matrix R, enclosed inside a simple closed curve γ running in the anti-clockwise direction, the projection P onto the eigenspace* $\ker(\lambda I - R)$ *is given by*

$$P = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (zI-R)^{-1} \mathrm{d}z.$$

We are now ready to give the **eighth proof** of Theorem 4.

Proof: Let $R = (r_{ij}) \in$ SO(3,ℝ). Again we restrict ourselves to the case that $R \neq I$. Then we have that one is an isolated simple eigenvalue. Let the other two eigenvalues be denoted by $\lambda, \overline{\lambda}$, and let $p_{ij}(z)$ be the minor obtained by deleting the row i and column j from the matrix $zI - R$. If γ encloses 1, but not the other two eigenvalues $\lambda, \overline{\lambda}$, then we have

$$\begin{aligned} P &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma (zI-R)^{-1} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{\det(zI-R)} (p_{ij}(z)) \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{(z-1)(z-\lambda)(z-\overline{\lambda})} (p_{ij}(z)) \mathrm{d}z = \frac{1}{(1-\lambda)(1-\overline{\lambda})} (p_{ij}(1)) \end{aligned}$$

where we have used the Cauchy Integral Formula [7, Corollary 3.5, page 94] to obtain the last equality. In particular

$$
\begin{aligned}
P\begin{pmatrix}1\\0\\0\end{pmatrix} &= \frac{1}{|1-\lambda|^2}\begin{pmatrix}(1-r_{22})(1-r_{33})-r_{23}r_{32}\\ r_{12}(1-r_{33})+r_{13}r_{32}\\ r_{12}r_{23}+r_{13}(1-r_{22})\end{pmatrix}\\
&= \frac{1}{|1-\lambda|^2}\begin{pmatrix}1-r_{22}-r_{33}+R_{11}\\ r_{12}+R_{21}\\ r_{13}+R_{31}\end{pmatrix}\\
&= \frac{1}{|1-\lambda|^2}\begin{pmatrix}1-r_{22}-r_{33}+r_{11}\\ r_{12}+r_{21}\\ r_{13}+r_{31}\end{pmatrix}\\
&= \frac{1}{|1-\lambda|^2}\begin{pmatrix}\dfrac{(r_{12}+r_{21})(r_{13}+r_{31})}{r_{23}+r_{32}}\\ r_{12}+r_{21}\\ r_{13}+r_{31}\end{pmatrix} = c\begin{pmatrix}\dfrac{1}{r_{23}+r_{32}}\\ \dfrac{1}{r_{13}+r_{31}}\\ \dfrac{1}{r_{12}+r_{21}}\end{pmatrix}
\end{aligned}
$$

for some constant c. ■

Note that we recover the vector W_1 from Theorem 3. W_2, W_3 can be found similarly.

9. What About Zeros?

Now it is time to think about the conditions $r_{ij} + r_{ji} \neq 0$. What if some of them failed e.g. $r_{12} + r_{21} = 0$? The eigenvector still exists, but how does it look now? Note first that

$$r_{13}^2 = 1 - r_{11}^2 - r_{12}^2 = 1 - r_{11}^2 - r_{21}^2 = r_{31}^2 \Rightarrow r_{13} = \pm r_{31}.$$

Similarly $r_{23} = \pm r_{32}$. So our matrix looks now as

$$\begin{pmatrix} a & r & q\\ -r & b & p\\ \varepsilon q & \zeta p & c\end{pmatrix}$$

where $\varepsilon^2 = \zeta^2 = 1$. Suppose first that $pqr \neq 0$. The orthogonality conditions for the first two rows gives

$$-ar + br + pq = 0 \Leftrightarrow pq = r(a-b).$$

For the first two columns we get instead

$$ar - br + \varepsilon\zeta pq = 0 \Leftrightarrow \varepsilon\zeta pq = r(b-a)$$

thus $\varepsilon\zeta = -1 \Leftrightarrow \zeta = -\varepsilon$.

Now for $\varepsilon = -1$ we simply put $V = (0 \;\; q \;\; -r)^t$. We have

$$RV = \begin{pmatrix} a & r & q \\ -r & b & p \\ -q & p & c \end{pmatrix} \begin{pmatrix} 0 \\ q \\ -r \end{pmatrix} = \begin{pmatrix} 0 \\ bq - rp \\ pq - cr \end{pmatrix} = \begin{pmatrix} 0 \\ q \\ -r \end{pmatrix}$$

where the last equality follows from Theorem 5.

If $\varepsilon = 1$ we take instead $V = (p \;\; 0 \;\; r)^t$ with similar argument

$$RV = \begin{pmatrix} a & r & q \\ -r & b & p \\ q & -p & c \end{pmatrix} \begin{pmatrix} p \\ 0 \\ r \end{pmatrix} = \begin{pmatrix} ap + qr \\ 0 \\ pq + cr \end{pmatrix} = \begin{pmatrix} p \\ 0 \\ r \end{pmatrix}$$

where again the last equality follows from Theorem 5.

Thus the rule is easy: for exactly one pair of indices i, j we have $r_{ij} = r_{ji}$. If k is the remaining index put $v_k = 0, v_i = r_{kj}, v_j = -r_{ki}$.

In fact we can describe matrices above almost explicitly. To make calculations more homogeneous we put $c = \varepsilon d$ as well. Consider the remaining orthogonal conditions for different rows

$$\begin{aligned} \varepsilon aq - \varepsilon pr + \varepsilon dq &= 0 \Leftrightarrow pr = q(a+d) \\ -\varepsilon pq - \varepsilon br + \varepsilon dr &= 0 \Leftrightarrow pq = r(-b+d). \end{aligned}$$

Pairwise multiplications of the obtained equations and cancelling gives

$$r^2 = (a-b)(a+d), \quad q^2 = (a-b)(-b+d), \quad p^2 = (a+d)(-b+d).$$

Now the last orthogonality condition is

$$\begin{aligned} 1 &= a^2 + p^2 + q^2 = a^2 + (-b+d)(2a - b + d) \\ &= a^2 + 2a(-b+d) + (-b+d)^2 = (a-b+d)^2 \end{aligned}$$

or $a - b + d = \pm 1$ (other rows and columns gives the same). Now we can choose a, b as parameters (with natural restrictions, e.g. $|a| < 1$) and reconstruct the rest choosing signs. As example we get

$$R = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix}.$$

It remains to consider the case $pqr = 0$. If for example $p = 0$ then by the orthogonality of two first rows $qr = 0$ as well and similarly for other cases we get that at least two of p, q, r are zero. Then the corresponding column containing them is an eigenvector directly.

10. Possible Generalisations

So far we concentrated on 3×3 real matrices, especially on the case $R \in$ SO(3,ℝ). But we now ask: what can be generalised? Theorem 4 is obviously valid for any orthogonal matrix (that is why we have $R \in$ O(3,ℝ) in the abstract), and moreover, it is valid for any matrix $R = cR'$ with $R' \in$ SO(3,ℝ) Theorem 3 is valid as well if we replace the constant 1 in the vectors W_i by $c \neq 0$.

For larger sizes, we still have the analogues of Theorem 2 and Theorem 5 and can imitate the second proof to obtain the analogues of the vectors W_i. But already for the size 5 (where the vector V with $RV = V$ exists), the expressions involve determinants of size 3, and its is hardly attractive to write them here. The vector U obtained in the third proof is also in principle available, but we have no easy analogue of Theorem 1, while an analogue of Theorem 2 produces the determinants of high order. And the idea to generalise Theorem 4 to higher dimensions looks hopeless.

What if we change the field? Because the conditions $R^{-1} = R^t$ and $\det R = 1$ are purely algebraic, all purely algebraic proofs survive, and we have the same Theorem 3 but we need some modifications.

First of all, we should understand why 1 is still an eigenvalue. This is easy. If α, β, γ are our eigenvalues, then $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is the same set of numbers, but they may be in a different order. If for example, $\frac{1}{\alpha} = \beta$ then $\alpha\beta = 1$ and the condition $\det R = 1$ gives $\gamma = 1$. The only remaining case is $\frac{1}{\alpha} = \alpha$, and then $\alpha = \pm 1$, and similarly for β and γ, but because their product is one, at least one of them is equal to one as well. So the second proof survives completely, and the third need only an adjustment in the place where we used Theorem 7.

The first proof has another weak point: for arbitrary field $x^2 + y^2 = 0$ does not imply $x = y = 0$ which we have used in the special case $R_{11} = -1$. The case when $r_{12} \neq 0$ can really happen. Here is a a nice example in $\mathbb{Z}_5$

$$R = \begin{pmatrix} -1 & -1 & -2 \\ -2 & -1 & -1 \\ -1 & -2 & -1 \end{pmatrix}.$$

But R still has a correct eigenvector. The proof therefore should be modified (e.g. consider i in our field such that $\mathrm{i}^2 = -1$, write $r_{13} = \mathrm{i} r_{12}$ and $r_{31} = \pm \mathrm{i} r_{21}$ and continue in the same style as we have done in the previous section to describe all possible exceptional matrices), but we prefer to skip this and restrict ourselves by only one algebraic proof.

So the conditions $R^{-1} = R^t$ and $\det R = 1$ are sufficient to our main theorems. The interesting question is therefore: what is the class of the matrices that satisfy those conditions? It is obviously a group. We study matrices of size 2 first.

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = R^{-1} = R^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

therefore $a = d$, $c = -b$, and $a^2 + b^2 = 1$. For the complex numbers, we put $a = \cos z, b = \sin z$ for some complex number z and get all the solutions. So matrices such as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos z & \sin z \\ 0 & -\sin z & \cos z \end{pmatrix}, \qquad \begin{pmatrix} \cos z & 0 & \sin z \\ 0 & 1 & 0 \\ -\sin z & 0 & \cos z \end{pmatrix}$$

and their products belongs to our group, so it is large enough. For finite fields we can have difficulties to find "cosines" (for example, in $\mathbb{Z}_5$, we have $a^2 + b^2 = 1 \Rightarrow a = 0, b = 1$ or $a = 1, b = 0$), but already in $\mathbb{Z}_7$ we have $2^2 + 2^2 = 1$ which produces some matrices. But we prefer to skip this intriguing topic for now.

Any time one gets a result about the orthogonal matrices, it is natural to wonder about their complex relatives - unitary matrices. What can be said about them? Most parts of the proofs fail, which is not surprising, because now $R_{ij} = \overline{r}_{ij}$, and skew-Hermitian matrix can be invertible, and can have non-zero elements on the main diagonal. So we have no direct analogue of Theorem 4. We can get some results if we know the eigenvalue, but is nothing else than the direct application of Theorem 2 (as in the second proof).

Theorem 12 *Let $R \in \mathrm{SU}(3)$ be an unitary matrix with (simple) eigenvalue equal to λ. Then for all the vectors*

$$\begin{aligned} W_1 &= \left(\overline{r_{11}} + \lambda^2 - \lambda(r_{22} + r_{33}),\ \overline{r_{12}} + r_{21},\ \overline{r_{13}} + r_{31} \right)^t \\ W_2 &= \left(r_{12} + r_{21},\ \overline{r_{22}} + \lambda^2 - \lambda(r_{11} - r_{33}),\ r_{23} + r_{32} \right)^t \\ W_3 &= \left(r_{13} + r_{31},\ r_{23} + r_{32},\ \overline{r_{33}} + \lambda^2 - \lambda(r_{11} - r_{22}) \right)^t \end{aligned}$$

we have $RW_i = \lambda W_i$, and at least one of them is non-zero, and therefore is the eigenvector.

Another possibility for generalisation suggested by the referee is to consider higher dimensional representations of the group $\mathrm{SO}(3, \mathbb{R})$ along the lines of investigations done in the articles [3] and [4]. We leave this interesting approach for future study.

11. Conclusions

In this article, we gave eight different elementary proofs for the existence of an eigenvector of a rotation matrix written explicitly in terms of the the entries of the matrix. Moreover, we also studied separately the pathological case when the eigenvector formula breaks down, and discussed various possible generalisations.

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