A SUMMATION METHOD BASED ON
THE FOURIER SERIES OF PERIODIC DISTRIBUTIONS
AND AN EXAMPLE ARISING IN THE CASIMIR EFFECT

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ABSTRACT. A generalised summation method is considered based on the Fourier series of periodic distributions. It is shown that

$$e^{it} - 2e^{2it} + 3e^{3it} - 4e^{4it} + - \cdots = \mathrm{Pf}\frac{e^{it}}{(1 + e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta'_{(2n+1)\pi},$$

where $\mathrm{Pf}\dfrac{e^{it}}{(1 + e^{it})^2} \in \mathcal{D}'(\mathbb{R})$ is the 2π-periodic distribution given by

$$\left\langle \mathrm{Pf}\frac{e^{it}}{(1 + e^{it})^2}, \varphi \right\rangle$$
$$= \sum_{n \in \mathbb{Z}} \int_0^{2\pi} \frac{(t - \pi)^2 e^{it}}{(1 + e^{it})^2} \int_0^1 (1 - \theta)\varphi''((2n + 1)\pi + \theta(t - \pi))d\theta dt,$$

for $\varphi \in \mathcal{D}(\mathbb{R})$. Applying the generalised summation method, the sum of the divergent series $1 + 2 + 3 + \cdots$ (arising in a quantum field theory calculation in the Casimir effect) is determined, and more generally also the sum $1^k + 2^k + 3^k + \cdots$, for $k \in \mathbb{N}$, is determined.

1. INTRODUCTION

A quantum field theoretic calculation predicts the so-called Casimir effect, giving the correct[1] value of the attractive force between two parallel perfect conductor plates in vacuum, if one uses the absurd sum

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}. \tag{1}$$

While there exists a generalised summation method (zeta function regularisation) that allows one to show this, it is somewhat contrived in the context of the Casimir effect calculation. On the other hand, Fourier series makes a direct appearance in the quantum field calculation (see e.g. [7, §III]), since one uses the quantised momentum space. We give an alternative summation method, based on distributional Fourier series, which gives a derivation of (1), and gives a rigorous mathematical justification of (1).

[1]by now experimentally verified [4]

A justification for

$$1 + 2 + 3 + \cdots = -\frac{1}{12},$$

for example[2] given by Ramanujan in one of his notebooks [1, Ch.6, p.135], is as follows:

If $s := 1 + 2 + 3 + \cdots$, then formally

$$
\begin{array}{rl}
s & = \quad 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\
-4s & = \quad\quad -4 \quad\quad -8 \quad -12\cdots \\
\hline
-3s & = \quad 1 - 2 + 3 - 4 + 5 - 6 + - \cdots .
\end{array}
\tag{2}
$$

The power series expansion

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + - \cdots \text{ for } |x| < 1$$

motivates associating the sum of the alternating series $1 - 2 + 3 - 4 + - \cdots$ with

$$\left.\frac{1}{(1+x)^2}\right|_{x=1},$$

and one writes

$$1 - 2 + 3 - 4 + - \cdots = \frac{1}{4}. \tag{3}$$

In light of the last equation from (2), together with (3), we arrive at

$$s = 1 + 2 + 3 + \cdots = -\frac{1}{3} \cdot \frac{1}{4} = -\frac{1}{12}. \tag{4}$$

There exist generalised summation techniques which allow one to obtain the result (4), for example the zeta function regularisation method [16, p.301]. We briefly recall the idea behind the zeta function regularisation method in Appendix B, in order to contrast it with our summation method introduced here.

In this article, we give an alternative route to obtaining (4), based on the Fourier series of periodic distributions. Distributional Fourier series converge in the sense of distributions whenever the Fourier coefficients c_n grow at most polynomially, that is, they satisfy for some $M > 0$ and $k > 0$ that

$$\text{for all } n \in \mathbb{Z}, \ |c_n| \leqslant M(1 + |n|)^k,$$

and so one could try making the above manipulations (2) with divergent series on a firmer footing by using the Fourier theory of distributions, where

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[2]There are older justifications of this. For example, in [12], the determination of $\zeta(-1)$ is attributed to Euler. The related summation of the divergent series $1 - 2 + 3 - + \cdots$, being assigned the sum $1/4$, can be based on the so-called *Euler summation method*. (For a modern discussion of the Euler summation method, see for example [8, §20, p.325-328].) Euler had discovered a functional equation relating the zeta function with the Dirichlet eta function (alternating zeta function) for integral values, which yields, using the alternating sum $1 - 2 + 3 - + \cdots = 1/4$, also that $\zeta(-1) = 1 + 2 + 3 + \cdots = -1/12$ [5, Vol. 14, p. 442-443, 594-595; Vol. 15, p. 70-90]. The determination of the zeta function at all negative integers, using the Euler summation method, can be found in [12].

the place holders of the terms of the divergent series are naturally provided by the characters e^{int}, and the operation in the second step of (2) can be carried out by simply doubling the period, that is, by a homothetic dilation by 2 of the distributional Fourier series. The question then arises if one can give an explicit formula for the distribution corresponding to the Fourier series (corresponding to the right hand side of the last line in (2))

$$e^{it} - 2e^{2it} + 3e^{3it} - 4e^{4it} + - \cdots.$$

The aim of this article is mainly to carry out this computation, and we give an explicit expression for this distribution. We also give an operation which can be justifiably thought of as being the operation of 'setting $t = 0$' in such a distributional Fourier series. Within the framework of our generalised summation method, this allows us to arrive at

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}.$$

We remark that this last series plays a role in the Casimir effect in quantum field theory [16, §6.6]. We briefly elaborate on this link in Appendix A.

The paper is organised as follows. We introduce the summation method in Section 2. In the Sections 3 and 4, we construct a certain 2π-periodic distribution S on $\mathbb{R}$, and in Section 5, we determine its Fourier coefficients. In Section 6, it is shown that S has a Fourier series that is summable at $t = 0$. Finally, in Sections 7 and 8, we apply our generalised summation method to determine $1 + 2 + 3 + \cdots$, and more generally, $1^k + 2^k + 3^k + \cdots$ for any $k \in \mathbb{N}$.

2. Summation method

For the background on the Fourier series theory of periodic distributions, we refer the reader to [10, Chap.IV], [17, p.91,101-104] and [3, Chap.33]. Throughout this article, unless otherwise indicated, we will use the standard distribution theory notation from Schwartz [11] or Tréves [15]. We recall the basic definitions below.

Definition 2.1 (Translation operator $\mathbf{T}_a$; periodic distributions).
Let $a \in \mathbb{R}$.

- For $f : \mathbb{R} \to \mathbb{R}$, $\mathbf{T}_a f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(\mathbf{T}_a f)(t) = f(t - a), \quad t \in \mathbb{R}.$$

- For $T \in \mathcal{D}'(\mathbb{R})$, $\mathbf{T}_a T \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle \mathbf{T}_a T, \varphi \rangle = \langle T, \mathbf{T}_{-a} \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

- Let $\tau > 0$. A distribution $T \in \mathcal{D}'(\mathbb{R})$ is said to be τ-*periodic* if $\mathbf{T}_\tau T = T$.

For computational ease, we will now work with a 2π-periodic distribution. A 2π-periodic distribution $T \in \mathcal{D}'(\mathbb{R})$ possesses a Fourier series expansion

$$T = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n(T)e^{int} =: \sum_{n \in \mathbb{Z}} c_n(T)e^{int},$$

where the $c_n(T)$ are complex numbers, and the convergence is in $\mathcal{D}'(\mathbb{R})$. The Fourier coefficients $c_n(T)$ are best expressed in terms of the distribution $T_{\text{circle}} \in \mathcal{D}'(\mathbb{T})$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, obtained by 'wrapping T on the circle'. We make this precise below.

For a $\varphi \in \mathcal{D}(\mathbb{R})$, we first define the 2π-periodic smooth function φ_{circle}

$$\varphi_{\text{circle}}(e^{it}) = \sum_{n \in \mathbb{Z}} \varphi(t - 2\pi n), \quad t \in \mathbb{R}.$$

We note that at each point $t \in \mathbb{R}$, only finitely many terms in the series above are nonzero. Then given a $T_{\text{circle}} \in \mathcal{D}'(\mathbb{T})$, this defines a 2π-periodic distribution $T \in \mathcal{D}'(\mathbb{R})$ as follows:

$$\langle T, \varphi \rangle = \langle T_{\text{circle}}, \varphi_{\text{circle}} \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

It can be shown that this map is a bijection between $\mathcal{D}'(\mathbb{T})$ and the subspace of $\mathcal{D}'(\mathbb{R})$ consisting of all 2π-periodic distributions; see [10, p.150-151]. The Fourier coefficients $c_n(T)$ are given by

$$c_n(T) = \frac{1}{2\pi}\langle T_{\text{circle}}, e^{-int} \rangle, \quad n \in \mathbb{Z}.$$

We now consider a sequence of pointwise nonnegative test functions $(\varphi_m)_{m \in \mathbb{N}}$, with shrinking supports, and unit area, which provides an approximation to the Dirac delta distribution δ_0 with support at 0.

Definition 2.2 (Symmetric positive mollifier, approximate identity)**.**
A test function $\varphi \in \mathcal{D}(\mathbb{R})$ is said to be a *symmetric positive mollifier* if

- $\varphi(t) = \varphi(-t)$ for all $t \in \mathbb{R}$,

- $\varphi(t) \geqslant 0$ for all $t \in \mathbb{R}$,

- $\displaystyle\int_{\mathbb{R}} \varphi(t)dt = 1.$

Define the sequence $(\varphi_m)_{m \in \mathbb{N}}$ by

$$\varphi_m(t) = m\varphi(mt), \quad t \in \mathbb{R}, \ \ m \in \mathbb{N}.$$

Then $\varphi_m \to \delta_0$ as $m \to \infty$ in $\mathcal{D}'(\mathbb{R})$, where for $a \in \mathbb{R}$ the notation δ_a is used for the Dirac distribution on $\mathbb{R}$ with support $\{a\}$.

We call $(\varphi_m)_{m \in \mathbb{N}}$ an *approximate identity*.

For a 2π-periodic distribution T, we have

$$T = \sum_{n\in\mathbb{Z}} c_n(T)e^{int},$$

and since this is an equality of distributions, we cannot in general 'set $t = 0$'. However, we could look at an approximate identity $(\varphi_m)_{m\in\mathbb{N}}$, and the limit

$$\lim_{m\to\infty}\langle T, \varphi_m\rangle,$$

if it exists, can be thought of as 'setting $t = 0$ in T'. For example, we have the following elementary result.

Proposition 2.3. *Let $f \in C(\mathbb{R}\backslash\{0\})$ be such that the limits*

$$f(0+) := \lim_{x\searrow 0} f(x) \text{ and } f(0-) := \lim_{x\nearrow 0} f(x)$$

exist, and let $T_f \in \mathcal{D}'(\mathbb{R})$ be the distribution given by

$$\langle T_f, \psi\rangle := \int_{\mathbb{R}} f(t)\psi(t)dt \text{ for } \psi \in \mathcal{D}(\mathbb{R}).$$

Then for any approximate identity $(\varphi_m)_{m\in\mathbb{N}}$, we have

$$\lim_{m\to\infty}\langle T_f, \varphi_m\rangle = \frac{f(0+) + f(0-)}{2}.$$

Proof. We have

$$
\begin{aligned}
\langle T_f, \varphi_m\rangle &= \int_{\mathbb{R}} f(t)\varphi_m(t)dt \\
&= \int_{-\infty}^{\infty} f(t)m\varphi(mt)dt \\
&= \int_{-\infty}^{\infty} f\left(\frac{\tau}{m}\right)\varphi(\tau)d\tau \\
&= \int_{-\infty}^{0} f\left(\frac{\tau}{m}\right)\varphi(\tau)d\tau + \int_{0}^{\infty} f\left(\frac{\tau}{m}\right)\varphi(\tau)d\tau.
\end{aligned}
$$

By the Lebesgue dominated convergence theorem, it follows that

$$
\begin{aligned}
\lim_{m\to\infty}\langle T_f, \varphi_m\rangle &= \int_{-\infty}^{0} f(0+)\varphi(\tau)d\tau + \int_{0}^{\infty} f(0-)\varphi(\tau)d\tau \\
&= f(0+)\int_{-\infty}^{0}\varphi(\tau)d\tau + f(0-)\int_{0}^{\infty}\varphi(\tau)d\tau \\
&= f(0+)\cdot\frac{1}{2}\int_{\mathbb{R}}\varphi(\tau)d\tau + f(0-)\cdot\frac{1}{2}\int_{\mathbb{R}}\varphi(\tau)d\tau \\
&= \frac{f(0+) + f(0-)}{2},
\end{aligned}
$$

where we used the fact that φ is a symmetric positive mollifier. $\square$

Definition 2.4.
For a 2π-periodic distribution T, let

$$\sum_{n\in\mathbb{Z}} c_n(T) e^{int} = T$$

in $\mathcal{D}'(\mathbb{R})$.

If there exists a $\sigma \in \mathbb{C}$ such that for any approximate identity $(\varphi_m)_{m\in\mathbb{N}}$, the limit $\lim_{m\to\infty} \langle T, \varphi_m \rangle$ exists, and $\lim_{m\to\infty} \langle T, \varphi_m \rangle = \sigma$, then we say *the series*

$$\sum_{n\in\mathbb{Z}} c_n(T)$$

is summable, or the *Fourier series of T is summable at $t = 0$*, and we define

$$\sum_{n\in\mathbb{Z}} c_n(T) = \sigma.$$

We call a double-sided complex sequence $(C_n)_{n\in\mathbb{Z}}$ *summable* if there exists a 2π-periodic distribution $T \in \mathcal{D}'(\mathbb{R})$ such that

- for all $n \in \mathbb{Z}$, $C_n = c_n(T)$, and

- the Fourier series of T is summable at $t = 0$.

(Since for any 2π-periodic distribution the Fourier coefficients are unique, this last definition is a well-defined notion.)

Notation 2.5 ($\boldsymbol{\sigma}$).
We denote the set of all 2π-periodic distributions that have a Fourier series that is summable at $t = 0$, in the sense of Definition 2.4, by $\boldsymbol{\sigma}$.

We have the following trivial observation that $\boldsymbol{\sigma}$ is a subspace of the space of all 2π-periodic distributions in $\mathcal{D}'(\mathbb{R})$.

Proposition 2.6. *If $T, S \in \mathcal{D}'(\mathbb{R})$ are 2π-periodic distributions belonging to $\boldsymbol{\sigma}$, then for any $\alpha, \beta \in \mathbb{C}$, also $\alpha T + \beta S$ also belongs to $\boldsymbol{\sigma}$, and moreover*

$$\sum_{n\in\mathbb{Z}} c_n(\alpha T + \beta S) = \alpha \sum_{n\in\mathbb{Z}} c_n(T) + \beta \sum_{n\in\mathbb{Z}} c_n(S).$$

Proof. This follows immediately from Definition 2.4 of summability and the linearity of the nth Fourier coefficient on the space of all 2π-periodic distributions. $\square$

Let us now show that $\boldsymbol{\sigma}$ is strictly contained in the subspace of all 2π-periodic distributions on $\mathbb{R}$.

Proposition 2.7. *Let T be the 2π-periodic distribution $T = \sum_{n\in\mathbb{Z}} e^{int}$.*

Then $T \notin \boldsymbol{\sigma}$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be a symmetric positive mollifier. Moreover, suppose that $\mathrm{supp}(\varphi) \subset (-2\pi, 2\pi)$ and $\varphi(0) > 0$. For $m \in \mathbb{N}$, with $\varphi_m := m\varphi(m\cdot)$, we have

$$
\begin{aligned}
\langle T, \varphi_m \rangle &= \sum_{n=-\infty}^{\infty} \langle e^{int}, \varphi_m \rangle \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{int} \varphi_m(t) dt \\
&= \sum_{n=-\infty}^{\infty} \widehat{\varphi_m}(-n),
\end{aligned}
\tag{5}
$$

where $\widehat{\varphi_m}$ denotes the Fourier transform of φ_m. By the Poisson summation formula for $\varphi_m \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ (where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of test functions), we have

$$
\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widehat{\varphi_m}(k) = \sum_{n=-\infty}^{\infty} \varphi_m(2\pi n).
\tag{6}
$$

Using (6) in (5), we obtain

$$
\begin{aligned}
\langle T, \varphi_m \rangle &= \sum_{n=-\infty}^{\infty} \widehat{\varphi_m}(-n) \\
&= \sum_{k=-\infty}^{\infty} \widehat{\varphi_m}(k) \qquad (\text{substituting } k := -n) \\
&= 2\pi \sum_{n=-\infty}^{\infty} \varphi_m(2\pi n) \\
&= 2\pi m \sum_{n=-\infty}^{\infty} \varphi(2\pi m n) \\
&= 2\pi m \varphi(0),
\end{aligned}
$$

where we have used the fact that $m \in \mathbb{N}$ and that $\mathrm{supp}(\varphi) \subset (-2\pi, 2\pi)$ in order to get the last equality. But then as $\varphi(0) > 0$, we have

$$
\lim_{m \to \infty} \langle T, \varphi_m \rangle = 2\pi \varphi(0) \cdot \lim_{m \to \infty} m = +\infty,
$$

and so $T \notin \boldsymbol{\sigma}$. $\qquad \square$

In order to 'insert zeroes' between the series terms (in our case between the Fourier series terms), as in the operation à la Ramanujan in (2) (when the $-4s$ sum is matched with the s sum in a particular manner), we will consider $\mathbf{H}_2 T$ for a periodic distribution T, where $\mathbf{H}_2 T$ is obtained from T by dilation by a factor of 2, hence 'doubling the period'.

Definition 2.8 (Homothetic transformation). Let $\lambda > 0$.

- For $f : \mathbb{R} \to \mathbb{R}$, $\mathbf{H}_\lambda f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(\mathbf{H}_\lambda f)(t) = f(\lambda t), \quad t \in \mathbb{R}.$$

- For $T \in \mathcal{D}'(\mathbb{R})$, $\mathbf{H}_\lambda T \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle \mathbf{H}_\lambda T, \varphi \rangle = \frac{1}{\lambda} \langle T, \mathbf{H}_{1/\lambda} \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Proposition 2.9. *Let T be a 2π-periodic distribution $T = \displaystyle\sum_{n \in \mathbb{Z}} c_n(T) e^{int}$. Then the 2π-periodic distribution $\mathbf{H}_2 T$ has the 'lacunary' Fourier series*

$$\mathbf{H}_2 T = \sum_{n \in \mathbb{Z}} c_n(T) e^{2int}.$$

Moreover, $\mathbf{H}_2 T \in \boldsymbol{\sigma}$ if and only if $T \in \boldsymbol{\sigma}$. Furthermore if $T \in \boldsymbol{\sigma}$, then we also have that

$$\sum_{n \in \mathbb{Z}} c_n(T) = \sum_{n \in \mathbb{Z}} c_n(\mathbf{H}_2 T).$$

Proof. The map $\mathbf{H}_\lambda : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is continuous, and so it follows by a termwise application of $\mathbf{H}_2$ on the Fourier series expansion of T that

$$\mathbf{H}_2 T = \sum_{n \in \mathbb{Z}} c_n(T) e^{2int}.$$

Now suppose that $(\varphi_m)_{m \in \mathbb{N}}$ is an approximate identity. Then we have that $\varphi_m = m\varphi(m\cdot)$, for a symmetric positive mollifier $\varphi \in \mathcal{D}(\mathbb{R})$. But then $\psi \in \mathcal{D}(\mathbb{R})$, given by

$$\psi = \frac{1}{2} \mathbf{H}_{\frac{1}{2}} \varphi,$$

is also a symmetric positive mollifier. Hence $(\psi_m)_{m \in \mathbb{N}}$, where $\psi_m = m\psi(m\cdot)$, is also an approximate identity.

If $T \in \boldsymbol{\sigma}$, then

$$\sum_{n \in \mathbb{Z}} c_n(T) = \lim_{m \to \infty} \langle T, \psi_m \rangle = \lim_{m \to \infty} \frac{1}{2} \langle T, \mathbf{H}_{\frac{1}{2}} \varphi_m \rangle = \lim_{m \to \infty} \langle \mathbf{H}_2 T, \varphi_m \rangle, \quad (7)$$

and so $\mathbf{H}_2 T \in \boldsymbol{\sigma}$ too, and moreover,

$$\sum_{n \in \mathbb{Z}} c_n(T) = \sum_{n \in \mathbb{Z}} c_n(\mathbf{H}_2 T). \quad (8)$$

Now suppose that $\mathbf{H}_2 T \in \boldsymbol{\sigma}$. Suppose that $(\psi_m)_{m \in \mathbb{N}}$ is an approximate identity. Then $\psi_m = m\psi(m\cdot)$, for a symmetric positive mollifier $\psi \in \mathcal{D}(\mathbb{R})$. But then $\varphi \in \mathcal{D}(\mathbb{R})$, given by

$$\varphi = 2\mathbf{H}_2 \psi,$$

is also a symmetric positive mollifier. Hence $(\varphi_m)_{m \in \mathbb{N}}$, where $\varphi_m = m\varphi(m\cdot)$, is also an approximate identity. Also,

$$\frac{1}{2} \mathbf{H}_{\frac{1}{2}} \varphi_m = \frac{m}{2} \varphi\left(\frac{m}{2}\cdot\right) = \frac{m}{2} 2(\mathbf{H}_2 \psi)\left(\frac{m}{2}\cdot\right) = m\psi\left(2\frac{m}{2}\cdot\right) = m\psi(m\cdot) = \psi_m.$$

Thus,

$$\sum_{n\in\mathbb{Z}} c_n(\mathbf{H}_2 T) = \lim_{m\to\infty} \langle \mathbf{H}_2 T, \varphi_m \rangle = \lim_{m\to\infty} \frac{1}{2}\langle T, \mathbf{H}_{\frac{1}{2}} \varphi_m \rangle = \lim_{m\to\infty} \langle T, \psi_m \rangle.$$

Consequently, $T \in \boldsymbol{\sigma}$, and again (8) holds. $\qquad\square$

Let $T \in \mathcal{D}'(\mathbb{R})$ be such that there exists a $k \neq 1$ such that $T - k\mathbf{H}_2 T$ has summable Fourier coefficients at $t = 0$ in the sense of Definition 2.4, that is,

$$\sum_{n\in\mathbb{Z}} c_n(T - k\mathbf{H}_2 T)$$

is summable. Then we have two possible cases:

$\underline{1^\circ}$ $T \in \boldsymbol{\sigma}$. Then it follows from Propositions 2.6 and 2.9 that

$$\sum_{n\in\mathbb{Z}} c_n(T) = \frac{1}{1-k} \sum_{n\in\mathbb{Z}} c_n(T - k\mathbf{H}_2 T).$$

$\underline{2^\circ}$ $T \notin \boldsymbol{\sigma}$, that is, T has a Fourier series that is not summable at $t = 0$ in the sense of Definition 2.4. Nevertheless, in light of 1° above, it is now natural to define

$$\sum_{n\in\mathbb{Z}} c_n(T) = \frac{1}{1-k} \sum_{n\in\mathbb{Z}} c_n(T - k\mathbf{H}_2 T).$$

In order to make this a formal definition, we need to check well-definedness, which is done below.

Proposition 2.10. *Suppose that $T \in \mathcal{D}'(\mathbb{R})$ is a 2π-periodic distribution, such that $T \notin \boldsymbol{\sigma}$, but there exist $\alpha, \beta \in \mathbb{C}\backslash\{1\}$ such that $T - \alpha\mathbf{H}_2 T \in \boldsymbol{\sigma}$ and $T - \beta\mathbf{H}_2 T \in \boldsymbol{\sigma}$. Then $\alpha = \beta$.*

Proof. Suppose that $\alpha \neq \beta$. As $T - \alpha\mathbf{H}_2 T \in \boldsymbol{\sigma}$ and $T - \beta\mathbf{H}_2 T \in \boldsymbol{\sigma}$, we have that for any approximate identity $(\varphi_m)_{m\in\mathbb{N}}$, both the limits

$$\sigma_\alpha := \lim_{m\to\infty} (\langle T, \varphi_m \rangle - \alpha\langle \mathbf{H}_2 T, \varphi \rangle), \tag{9}$$

$$\sigma_\beta := \lim_{m\to\infty} (\langle T, \varphi_m \rangle - \beta\langle \mathbf{H}_2 T, \varphi \rangle) \tag{10}$$

exist. But then taking the difference of (9) and (10) gives

$$\sigma_\alpha - \sigma_\beta = \lim_{m\to\infty} (\beta - \alpha)\langle \mathbf{H}_2 T, \varphi \rangle,$$

and as $\alpha \neq \beta$,

$$\lim_{m\to\infty} \langle \mathbf{H}_2 T, \varphi \rangle = \frac{1}{\beta - \alpha}(\sigma_\alpha - \sigma_\beta)$$

exists. But then, adding α times $\lim_{m\to\infty} \langle \mathbf{H}_2 T, \varphi \rangle$ to (9), gives

$$\lim_{m\to\infty} \langle T, \varphi_m \rangle = \lim_{m\to\infty} (\langle T, \varphi_m \rangle - \alpha\langle \mathbf{H}_2 T, \varphi \rangle) + \alpha \lim_{m\to\infty} \langle \mathbf{H}_2 T, \varphi \rangle$$

exists, that is, $T \in \boldsymbol{\sigma}$, a contradiction. Hence $\alpha = \beta$. $\qquad\square$

Proposition 2.10 takes care of the well-definition in 2° on page 9, and we now have the following extension of the summation method for Fourier coefficients given in Definition 2.4, from $\boldsymbol{\sigma}$ to the larger set $\boldsymbol{\Sigma}$, defined below.

Notation 2.11 ($\boldsymbol{\Sigma}$).
We set $\boldsymbol{\Sigma} := \{T \in \mathcal{D}'(\mathbb{R}) : \exists k \neq 1 \text{ such that } T - k\mathbf{H}_2 T \in \boldsymbol{\sigma}\}$.

Definition 2.12. Let $T \in \mathcal{D}'(\mathbb{R})$ be a 2π-periodic distribution T such that there exists a $k \in \mathbb{C}\backslash\{1\}$ such that $T - k\mathbf{H}_2 T \in \boldsymbol{\sigma}$. Then we say that *the series*

$$\sum_{n \in \mathbb{Z}} c_n(T)$$

is summable, or the *Fourier series of T is summable at $t = 0$*, and we define

$$\sum_{n \in \mathbb{Z}} c_n(T) = \frac{1}{1-k} \sum_{n \in \mathbb{Z}} c_n(T - k\mathbf{H}_2 T).$$

We call a double-sided complex sequence $(C_n)_{n \in \mathbb{Z}}$ *summable* if there exists a 2π-periodic distribution T such that

- for all $n \in \mathbb{Z}$, $C_n = c_n(T)$, and
- the Fourier series of T is summable at $t = 0$.

3. The distribution $\mathrm{Pf}\dfrac{e^{it}}{(1 + e^{it})^2} \in \mathcal{D}'((0, 2\pi))$

For $0 < \epsilon < \pi$, let $\Omega_\epsilon := (0, \pi - \epsilon) \cup (\pi + \epsilon, 2\pi)$.

For $\varphi \in \mathcal{D}((0, 2\pi))$, we define

$$\left\langle \mathrm{Pf}\frac{e^{it}}{(1 + e^{it})^2}, \varphi \right\rangle := \lim_{\epsilon \searrow 0} \left(\int_{\Omega_\epsilon} \frac{\varphi(t)e^{it}}{(1 + e^{it})^2}dt - \frac{\varphi(\pi)}{\tan(\epsilon/2)} \right). \qquad (11)$$

In the following result, we will give an alternative expression for the right hand side of (11), and show that $\mathrm{Pf}\dfrac{e^{it}}{(1 + e^{it})^2}$ defines a distribution on the open interval $(0, 2\pi)$.

Proposition 3.1. *For any $\varphi \in \mathcal{D}((0, 2\pi))$, we have*

$$\left\langle \mathrm{Pf}\frac{e^{it}}{(1 + e^{it})^2}, \varphi \right\rangle = \int_0^{2\pi} \frac{(t - \pi)^2 e^{it}}{(1 + e^{it})^2} \int_0^1 (1 - \theta)\varphi''(\pi + \theta(t - \pi))d\theta dt,$$

and $\mathrm{Pf}\dfrac{e^{it}}{(1 + e^{it})^2} \in \mathcal{D}'((0, 2\pi))$.

Proof. By Taylor's Formula [17, (e),p.45] we have

$$\varphi(t) = \varphi(\pi) + (t - \pi)\varphi'(\pi) + (t - \pi)^2 \int_0^1 (1 - \theta)\varphi''(\pi + \theta(t - \pi))d\theta.$$

We have

$$\int_{\Omega_\epsilon} \frac{\varphi(t)e^{it}}{(1+e^{it})^2}dt = \varphi(\pi)\underbrace{\int_{\Omega_\epsilon}\frac{e^{it}}{(1+e^{it})^2}dt}_{A} + \varphi'(\pi)\underbrace{\int_{\Omega_\epsilon}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt}_{B}$$

$$+\underbrace{\int_{\Omega_\epsilon}\frac{(t-\pi)^2e^{it}}{(1+e^{it})^2}\int_0^1(1-\theta)\varphi''(\pi+\theta(t-\pi))d\theta dt.}_{C}$$

We will treat the three summands A, B, C separately below. It can be seen that the second summand above vanishes, by splitting the integral B on $\Omega_\epsilon = (0, \pi-\epsilon) \cup (\pi+\epsilon, 2\pi)$ into two, and using the substitution $\tau = 2\pi - t$ for the integral on the domain $(\pi+\epsilon, 2\pi)$:
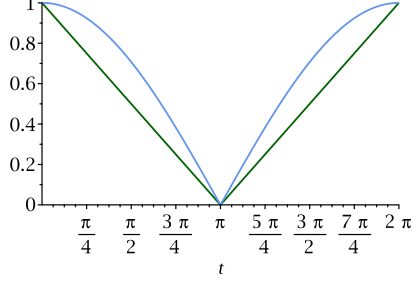
$$\begin{aligned}
B &= \int_{\Omega_\epsilon}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt \\
&= \int_0^{\pi-\epsilon}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt + \int_{\pi+\epsilon}^{2\pi}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt \\
&= \int_0^{\pi-\epsilon}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt + \int_{\pi-\epsilon}^0\frac{(\pi-\tau)e^{-i\tau}}{(1+e^{-i\tau})^2}(-1)d\tau \\
&= \int_0^{\pi-\epsilon}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt - \int_0^{\pi-\epsilon}\frac{(\tau-\pi)e^{-i\tau}}{(e^{i\tau}+1)^2e^{-2i\tau}}d\tau \\
&= \int_0^{\pi-\epsilon}\frac{(t-\pi)e^{it}}{(1+e^{it})^2}dt - \int_0^{\pi-\epsilon}\frac{(\tau-\pi)e^{i\tau}}{(1+e^{i\tau})^2}d\tau \\
&= 0.
\end{aligned}$$

In order to compute A, we note that $\dfrac{d}{dt}\dfrac{1}{1+e^{it}} = -\dfrac{ie^{it}}{(1+e^{it})^2}$, and so

$$\begin{aligned}
A = \int_{\Omega_\epsilon}\frac{e^{it}}{(1+e^{it})^2}dt &= \frac{i}{1+e^{it}}\Big|_0^{\pi-\epsilon} + \frac{i}{1+e^{it}}\Big|_{\pi+\epsilon}^{2\pi} \\
&= i\left(\frac{1}{1+e^{i(\pi-\epsilon)}} - \frac{1}{2} + \frac{1}{2} - \frac{1}{1+e^{i(\pi+\epsilon)}}\right) \\
&= i\left(\frac{e^{i\epsilon}}{e^{i\epsilon}-1} - \frac{1}{1-e^{i\epsilon}}\right) \\
&= i\frac{2\cos(\epsilon/2)}{2i\sin(\epsilon/2)} \\
&= \frac{1}{\tan(\epsilon/2)}.
\end{aligned}$$

Finally we consider the integral C, and we are interested in taking the limit of this integral as $\epsilon \searrow 0$.

For $t \in [0, 2\pi]$, $|\cos(t/2)| \geqslant \dfrac{|\pi - t|}{\pi}$.

Thus
$$\left| \frac{(t-\pi)^2 e^{it}}{(1+e^{it})^2} \right| = \frac{(t-\pi)^2}{|e^{it/2} + e^{-it/2}|^2} = \frac{(t-\pi)^2}{4(\cos(t/2))^2} \leqslant \frac{\pi^2}{4}.$$

So we have
$$\lim_{\epsilon \searrow 0} \int_{\Omega_\epsilon} \frac{(t-\pi)^2 e^{it}}{(1+e^{it})^2} \int_0^1 (1-\theta)\varphi''(\pi + \theta(t-\pi)) d\theta dt$$
$$= \int_0^{2\pi} \frac{(t-\pi)^2 e^{it}}{(1+e^{it})^2} \int_0^1 (1-\theta)\varphi''(\pi + \theta(t-\pi)) d\theta dt.$$

Hence
$$\left\langle \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2}, \varphi \right\rangle \ := \ \lim_{\epsilon \searrow 0} \left(\int_{\Omega_\epsilon} \frac{\varphi(t)e^{it}}{(1+e^{it})^2} dt - \frac{\varphi(\pi)}{\tan(\epsilon/2)} \right)$$
$$= \ \int_0^{2\pi} \frac{(t-\pi)^2 e^{it}}{(1+e^{it})^2} \int_0^1 (1-\theta)\varphi''(\pi + \theta(t-\pi)) d\theta dt.$$

The linearity of
$$\varphi \mapsto \left\langle \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2}, \varphi \right\rangle : \mathcal{D}((0,\pi)) \to \mathbb{C}$$

is obvious. In order to show continuity, we note that if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence that converges to 0 in $\mathcal{D}((0, 2\pi))$, then in particular, $(\varphi_n'')_{n\in\mathbb{N}}$ converges to 0 uniformly on $(0, 2\pi)$, and so

$$\left| \left\langle \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2}, \varphi_n \right\rangle \right| \ = \ \left| \int_0^{2\pi} \frac{(t-\pi)^2 e^{it}}{(1+e^{it})^2} \int_0^1 (1-\theta)\varphi_n''(\pi + \theta(t-\pi)) d\theta dt \right|$$
$$\leqslant \ 2\pi \cdot \frac{\pi^2}{4} \cdot 1 \cdot \|\varphi_n''\|_\infty \xrightarrow{n\to\infty} 0.$$

Consequently, $\mathrm{Pf}\dfrac{e^{it}}{(1+e^{it})^2} \in \mathcal{D}'((0, 2\pi))$. $\qquad\qquad\qquad\qquad \square$

4. 2π-PERIODIC EXTENSION

For $\varphi \in \mathcal{D}(\mathbb{R})$, we define the 2π-periodic extension of

$$\mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} \in \mathcal{D}'((0,2\pi))$$

to $\mathbb{R}$, which we will denote denoted by the same symbol. This can be done as follows: We note that as the function

$$\frac{e^{it}}{(1+e^{it})^2}$$

is continuous around 0 and 2π, and there is no problem extending our old distribution

$$\mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} \in \mathcal{D}'((0,2\pi))$$

to a slightly bigger open set $(0-\delta, 2\pi+\delta)$ for a small $\delta > 0$, so that in fact

$$\mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} \in \mathcal{D}'((0-\delta, 2\pi+\delta)).$$

Then the patching up of shifts of this distribution by integer multiples of 2π in order to define a global distribution on $\mathbb{R}$ can be done using the following principle of piecewise pasting/'recollement des morceaux' [11, Theorem IV, p.27] or [6, Theorem 1.4.3, p.16-17].

Proposition 4.1. *Let $\{\Omega_i\}_{i\in I}$ be an open cover of $\mathbb{R}$. Let $T_i \in \mathcal{D}'(\Omega_i)$ be such that whenever for $i \neq j$ we have $\Omega_i \cap \Omega_j \neq \varnothing$, then $T_i|_{\Omega_i \cap \Omega_j} = T_j|_{\Omega_i \cap \Omega_j}$. Then there exists a unique distribution $T \in \mathcal{D}'(\mathbb{R})$ such that $T|_{U_i} = T_i$ for all $i \in I$, given by*

$$\langle T, \varphi \rangle = \sum_{i \in I} \langle T_i, \varphi \alpha_i \rangle,$$

where $\{\alpha_i\}_{i\in I}$ is a locally finite partition of unity[3].

Notation 4.2 (The distribution S)**.**

We define the distribution $S \in \mathcal{D}'(\mathbb{R})$ by

$$S := \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta'_{(2n+1)\pi}.$$

It follows from the construction of S that S is 2π-periodic.

---

[3]That is, α_i, $i \in I$, are C^∞ functions such that $0 \leqslant \alpha_i \leqslant 1$, $E_i := \mathrm{supp}(\alpha_i) \subset \Omega_i$, $\{E_i\}_{i\in I}$ is locally finite (that is, for all $x \in \mathbb{R}$, there exists a neighbourhood of x that intersects only finitely many of the E_i, and $\sum_{i \in I} \alpha_i = 1$.

5. FOURIER COEFFICIENTS OF $S = \mathrm{Pf}\dfrac{e^{it}}{(1+e^{it})^2} + i\pi \displaystyle\sum_{n\in\mathbb{Z}} \delta'_{(2n+1)\pi}$

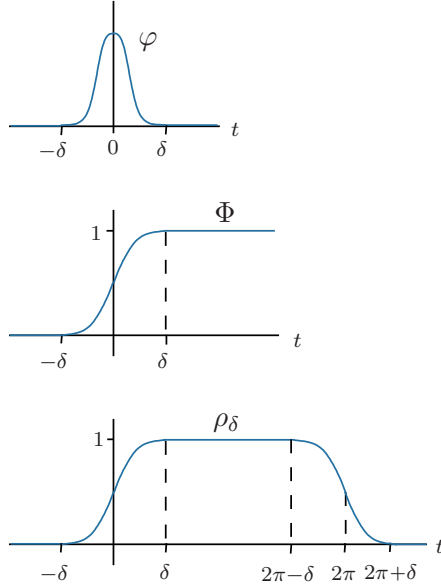**Theorem 5.1.** *The following Fourier series expansion is valid in $\mathcal{D}'(\mathbb{R})$:*

$$
\begin{aligned}
S = \mathrm{Pf}\dfrac{e^{it}}{(1+e^{it})^2} + i\pi \sum_{n\in\mathbb{Z}} \delta'_{(2n+1)\pi} \;&=\; e^{it} - 2e^{2it} + 3e^{3it} - 4e^{4it} + - \cdots \\
&=\; \sum_{n=1}^{\infty} (-1)^{n-1} n e^{nit}.
\end{aligned}
$$

Proof. We produce, for each $\delta > 0$, a test function $\rho_\delta \in \mathcal{D}(\mathbb{R})$ such that

$$
\rho_\delta\Big|_{(\delta,2\pi-\delta)} = 1,
$$

and such that we obtain a partition of unity on $\mathbb{R}$ by taking shifts of ρ_δ by integer multiples of 2π, that is,

$$
\sum_{n\in\mathbb{Z}} \rho_\delta(t + 2\pi n) = 1 \quad (t \in \mathbb{R}).
$$

Start with a $\delta > 0$ small, and consider any even, nonnegative test function φ with support in $[-\delta, \delta]$ and such that

$$
\int_{-\delta}^{\delta} \varphi(t)\,dt = 1.
$$

Define the function

$$
\Phi(t) := \int_{(-\infty,t]} \varphi(\tau)\,d\tau.
$$

Then it can be seen that for all $t \in \mathbb{R}$, Φ satisfies $\Phi(t) + \Phi(-t) = 1$. The desired ρ_δ can now be defined by

$$\rho_\delta(t) = \Phi(t) \cdot \Phi(2\pi - t).$$

Define $\varphi_\delta \in \mathcal{D}(\mathbb{R})$ by

$$\varphi_\delta(t) = \rho_\delta(t) e^{-int}.$$

Then we have

$$\varphi_{\delta,\text{circle}}(e^{it}) = \sum_{m \in \mathbb{Z}} \varphi_\delta(t - 2\pi m) = \sum_{m \in \mathbb{Z}} \rho_\delta(t - 2\pi m) e^{-int} = e^{-int}.$$

Hence

$$c_n(S) = \frac{1}{2\pi} \langle S_{\text{circle}}, e^{-int} \rangle = \frac{1}{2\pi} \langle S_{\text{circle}}, \varphi_{\delta,\text{circle}} \rangle = \frac{1}{2\pi} \langle S, \varphi_\delta \rangle.$$

Before proceeding, we explain the idea in the calculation below: We note that as $\delta \searrow 0$, φ_δ converges pointwise to 1 on $(0, 2\pi)$, and to 0 on $\mathbb{R} \backslash [0, 2\pi]$. So we expect $\langle S, \varphi_\delta \rangle$ to essentially be the action of S on e^{-int}. However, to show this, we need to work with the two limiting processes $\delta \searrow 0$ as well as $\epsilon \searrow 0$ (arising from the definition of S involving the distribution $\text{Pf}(\cdots)$). So we will split the set containing the support of φ_δ into two parts, one where the ϵ limit is relevant, but where ρ_δ is 'nice', namely a constant equal to 1, and the part where the δ limit is relevant, but the distribution is 'nice', namely it is a regular distribution[4].

We partition $O_\epsilon = \left(-\dfrac{\pi}{2}, \pi - \epsilon\right) \cup \left(\pi + \epsilon, 2\pi + \dfrac{\pi}{2}\right)$ as

$$O_\epsilon = \underbrace{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(3\frac{\pi}{2}, 2\pi + \frac{\pi}{2}\right)}_{V} \cup \underbrace{\left(\frac{\pi}{2}, \pi - \epsilon\right) \cup \left(\pi + \epsilon, 3\frac{\pi}{2}\right)}_{V_\epsilon}.$$

Then we have

$$
\left\langle \text{Pf} \frac{e^{it}}{(1 + e^{it})^2}, \varphi_\delta \right\rangle
$$

$$
= \lim_{\epsilon \searrow 0} \left(\int_{O_\epsilon} \frac{\varphi_\delta(t) e^{it}}{(1 + e^{it})^2} dt - \frac{e^{-in\pi} \cdot \rho_\delta(\pi)}{\tan(\epsilon/2)} \right)
$$

$$
= \int_V \frac{\varphi_\delta(t) e^{it}}{(1 + e^{it})^2} dt + \lim_{\epsilon \searrow 0} \left(\int_{V_\epsilon} \frac{1 \cdot e^{-int} e^{it}}{(1 + e^{it})^2} dt - \frac{(-1)^n}{\tan(\epsilon/2)} \right)
$$

$$
= \lim_{\delta \searrow 0} \int_V \frac{\rho_\delta(t) e^{-int} e^{it}}{(1 + e^{it})^2} dt + \lim_{\epsilon \searrow 0} \left(\int_{V_\epsilon} \frac{e^{-int} e^{it}}{(1 + e^{it})^2} dt - \frac{(-1)^n}{\tan(\epsilon/2)} \right)
$$

$$
= \int_V \frac{1 \cdot e^{-int} e^{it}}{(1 + e^{it})^2} dt + \lim_{\epsilon \searrow 0} \left(\int_{V_\epsilon} \frac{e^{-int} e^{it}}{(1 + e^{it})^2} dt - \frac{(-1)^n}{\tan(\epsilon/2)} \right)
$$

$$
= \lim_{\epsilon \searrow 0} \left(\int_{\Omega_\epsilon} \frac{e^{-int} e^{it}}{(1 + e^{it})^2} dt - \frac{(-1)^n}{\tan(\epsilon/2)} \right).
$$

---

[4]Throughout the article, by a regular distribution $T_f \in \mathcal{D}'(\mathbb{R})$ corresponding to a locally integrable function $f \in \mathrm{L}^1_{\text{loc}}(\mathbb{R})$, we mean the distribution $\mathcal{D}(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t) dt \in \mathbb{R}$.

In the above, we used the Lebesgue dominated convergence theorem in the evaluation of the δ-limit in order to get the second-last equality. Also,

$$
\left\langle i\pi \sum_{n\in\mathbb{Z}} \delta'_{(2n+1)\pi}, \varphi_\delta \right\rangle = -i\pi\varphi'_\delta(\pi)
$$

$$
= -i\pi(-in)e^{-in\pi} \cdot 1
$$

$$
= -(-1)^n n\pi.
$$

Hence

$$
c_n(S) = \frac{1}{2\pi}\langle S, \varphi_\delta \rangle
$$

$$
= \left\langle \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} + i\pi \sum_{n\in\mathbb{Z}} \delta'_{(2n+1)\pi}, \varphi_\delta \right\rangle
$$

$$
= \frac{1}{2\pi}\lim_{\epsilon\searrow0}\left(\int_{\Omega_\epsilon} \frac{e^{-int}e^{it}}{(1+e^{it})^2}dt - \frac{(-1)^n}{\tan(\epsilon/2)} - (-1)^n n\pi\right).
$$

Suppose that $\epsilon \in (0,\pi)$ is fixed. Let C_ϵ be the path along the circular arc $t \mapsto e^{it}$ for $t \in [\pi+\epsilon, 2\pi] \cup [0, \pi-\epsilon]$, traversed in the anticlockwise direction. The straight line segment L_ϵ is given by $t \mapsto -\cos\epsilon - it$, for $t \in [-\sin\epsilon, \sin\epsilon]$. Moreover, let C be the circular path with center 0 and radius $1/2$, traversed once in an anticlockwise manner.
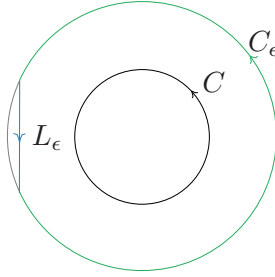
We have

$$
\frac{1}{2\pi}\int_{\Omega_\epsilon}\frac{e^{-int}e^{it}}{(1+e^{it})^2}dt = \frac{1}{2\pi i}\int_{\Omega_\epsilon}\frac{e^{-int}}{(1+e^{it})^2}ie^{it}dt
$$

$$
= \frac{1}{2\pi i}\int_{C_\epsilon}\frac{z^{-n}}{(1+z)^2}dz
$$

$$
= \frac{1}{2\pi i}\int_{C_\epsilon+L_\epsilon}\frac{z^{-n}}{(1+z)^2}dz - \frac{1}{2\pi i}\int_{L_\epsilon}\frac{z^{-n}}{(1+z)^2}dz
$$

$$
= \frac{1}{2\pi i}\oint_C\frac{z^{-n}}{(1+z)^2}dz - \frac{1}{2\pi i}\int_{L_\epsilon}\frac{z^{-n}}{(1+z)^2}dz.
$$

But by termwise differentiating the geometric series in the disk $|z| < 1$, we obtain

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 + - \cdots \quad (|z| < 1),$$

and so

$$\frac{z^{-n}}{(1+z)^2} = \frac{1}{z^n}(1 - 2z + 3z^2 - 4z^3 + - \cdots + (-1)^{n-1}nz^{n-1} + \cdots \quad (|z| < 1),$$

which yields, by the Laurent series theorem [9, Thm.4.7,p.134], that

$$\frac{1}{2\pi i} \oint_C \frac{z^{-n}}{(1+z)^2}dz = (-1)^{n-1}n =: d_n \text{ if } n = 1, 2, 3, \cdots.$$

Define

$$
\begin{aligned}
\Delta \quad &:= \quad c_n - d_n \\
&= \quad \frac{1}{2\pi}\lim_{\epsilon\searrow 0}\left(\int_{L_\epsilon}\frac{z^{-n}}{(1+z)^2}dz - \frac{(-1)^n}{\tan(\epsilon/2)} - (-1)^n n\pi\right).
\end{aligned}
$$

We have, using the fundamental theorem of contour integration, that

$$
\begin{aligned}
i\int_{L_\epsilon}\frac{1}{(1+z)^2}dz \quad &= \quad -i\int_{L_\epsilon}\frac{d}{dz}\frac{1}{1+z}dz \\
&= \quad -i\left(\frac{1}{1-e^{i\epsilon}} - \frac{1}{1-e^{-i\epsilon}}\right) \\
&= \quad i\frac{e^{i\epsilon}+1}{e^{i\epsilon}-1} = \frac{1}{\tan(\epsilon/2)}.
\end{aligned}
$$

So

$$
\begin{aligned}
\Delta \quad &= \quad \frac{1}{2\pi}\lim_{\epsilon\searrow 0}\left(\int_{L_\epsilon}\frac{z^{-n}}{(1+z)^2}dz - \int_{L_\epsilon}\frac{(-1)^n}{(1+z)^2}dz - (-1)^n n\pi\right) \\
&= \quad \frac{1}{2\pi}\lim_{\epsilon\searrow 0}\left(\int_{L_\epsilon}\frac{z^{-n}-(-1)^n}{(1+z)^2}dz - (-1)^n n\pi\right).
\end{aligned}
$$

If $n = 0$, then $\Delta = 0$. If $n \neq 0$, then we have, by a Taylor expansion of z^{-1} around the point $z = -1$, that

$$z^{-n} = (-1)^n + (-n)(-1)^{-n-1}(z+1) + h(z)$$

for $z \in L_\epsilon$, where $h(z) = O((z+1)^2)$ as $z \to -1$. Hence by the ML inequality,

$$\left|\int_{L_\epsilon}\frac{z^{-n}-(-1)^n}{(1+z)^2}dz - \int_{L_\epsilon}\frac{(-n)(-1)^{-n-1}}{1+z}dz\right| \sim \epsilon \xrightarrow{\epsilon\to 0} 0.$$

Hence

$$
\begin{aligned}
\Delta &= \frac{1}{2\pi}\lim_{\epsilon\searrow 0}\left(\int_{L_\epsilon}\frac{(-n)(-1)^{-n-1}}{1+z}dz - (-1)^n n\pi\right) \\
&= \frac{1}{2\pi}n(-1)^n\lim_{\epsilon\searrow 0}\left(i\int_{L_\epsilon}\frac{1}{1+z}dz - \pi\right).
\end{aligned}
$$

But

$$
\begin{aligned}
\int_{L_\epsilon}\frac{1}{1+z}dz &= \int_{\sin\epsilon}^{-\sin\epsilon}\frac{1}{1-\cos\epsilon+it}idt \\
&= i\int_{\sin\epsilon}^{-\sin\epsilon}\frac{1-\cos\epsilon-it}{(1-\cos\epsilon)^2+t^2}dt \\
&= i\int_{\sin\epsilon}^{-\sin\epsilon}\frac{1-\cos\epsilon}{(1-\cos\epsilon)^2+t^2}dt + 0 \\
&= -2i(1-\cos\epsilon)\int_0^{\sin\epsilon}\frac{1}{(1-\cos\epsilon)^2+t^2}dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_{L_\epsilon}\frac{1}{1+z}dz &= -2i(1-\cos\epsilon)\frac{1}{1-\cos\epsilon}\tan^{-1}\left(\frac{t}{1-\cos\epsilon}\right)\Big|_0^{\sin\epsilon} \\
&= -2i\tan^{-1}\frac{\sin\epsilon}{1-\cos\epsilon} \\
&= -2i\tan^{-1}\frac{\cos(\epsilon/2)}{\sin(\epsilon/2)} \\
&= -2i\left(\frac{\pi}{2}-\frac{\epsilon}{2}\right).
\end{aligned}
$$

So

$$
\begin{aligned}
\Delta &= \frac{1}{2\pi}n(-1)^n\left(i(-2i)\left(\frac{\pi}{2}-0\right)-\pi\right) \\
&= \frac{1}{2\pi}n(-1)^n(\pi-\pi) \\
&= 0.
\end{aligned}
$$

Consequently,

$$
\mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2}+i\pi\sum_{n\in\mathbb{Z}}\delta'_{(2n+1)\pi} = e^{it}-2e^{2it}+3e^{3it}-4e^{4it}+-\cdots.
$$

where the series on the right hand side converges in $\mathcal{D}'(\mathbb{R})$. $\square$

6. THE FOURIER SERIES OF S IS SUMMABLE AT $t = 0$

In this section, we will show that the 2π-distribution $S \in \mathcal{D}'(\mathbb{R})$, given in Notation 4.2 on page 13, belongs to $\boldsymbol{\sigma}$, that is, it has a Fourier series which is summable at $t = 0$ (in the sense of Definition 2.4).

Let $(\varphi_m)_{m \in \mathbb{N}}$ be any approximate identity. Then we have, for all large enough m, that the support of φ_m is contained inside, say $(-\pi/2, \pi/2)$, and so, in particular, it is far from $\pm \pi$. Thus if $U_\epsilon := (-\pi/2, \pi - \epsilon)$, then we have for all large m, that

$$\left\langle \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta'_{(2n+1)\pi}, \varphi_m \right\rangle = \lim_{\epsilon \searrow 0} \left(\int_{U_\epsilon} \frac{\varphi_m(t)e^{it}}{(1+e^{it})^2} dt - \frac{0}{\tan(\epsilon/2)} \right) + i\pi \cdot 0$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\varphi_m(t)e^{it}}{(1+e^{it})^2} dt.$$

Hence,

$$\lim_{m \to \infty} \langle S, \varphi_m \rangle = \lim_{m \to \infty} \left\langle \mathrm{Pf}\frac{e^{it}}{(1+e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta'_{(2n+1)\pi}, \varphi_m \right\rangle$$

$$= \lim_{m \to \infty} \int_{-\pi/2}^{\pi/2} \frac{\varphi_m(t)e^{it}}{(1+e^{it})^2} dt = \lim_{m \to \infty} \int_{-\pi/2}^{\pi/2} \frac{\varphi_m(t)e^{it}\psi(t)}{(1+e^{it})^2} dt,$$

where $\psi \in \mathcal{D}(\mathbb{R})$ is such that $\psi \equiv 1$ on $(-\pi/2, \pi/2)$ and has compact support contained inside $(-\pi, \pi)$. Thus

$$\lim_{m \to \infty} \langle S, \varphi_m \rangle = \lim_{m \to \infty} \left\langle \varphi_m, \frac{e^{it}\psi}{(1+e^{it})^2} \right\rangle \qquad (*)$$

$$= \frac{1 \cdot 1}{(1+1)^2} \qquad (\star)$$

$$= \frac{1}{4}.$$

In the first equality $(*)$ above, we view φ_m as the regular distribution in $\mathcal{E}'(\mathbb{R})$ corresponding to φ_m, and

$$t \mapsto \frac{e^{it}\psi(t)}{(1+e^{it})^2}$$

as a test function in $\mathcal{E}(\mathbb{R})$. Then, since

$$\lim_{m \to \infty} \varphi_m = \delta_0$$

in $\mathcal{D}'(\mathbb{R})$, the second equality $(\star)$ above follows.

Consequently, $S \in \boldsymbol{\sigma}$, that is, the distribution $S = \mathrm{Pf}\dfrac{e^{it}}{(1+e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta'_{(2n+1)\pi}$

has a Fourier series that is summable at $t = 0$, and

$$1 - 2 + 3 - 4 + \cdots = \sum_{n \in \mathbb{Z}} c_n(S) = \frac{1}{4}.$$

7. Ramanujan manipulation using Fourier series

Let $T_0 \in \mathcal{D}'(\mathbb{R})$ be the 2π-periodic distribution having the Fourier series

$$T_0 := e^{it} + 2e^{2it} + 3e^{3it} + \cdots. \tag{12}$$

The series converges in $\mathcal{D}'(\mathbb{R})$. We show in this section that $T_0 \in \mathbf{\Sigma}$, that is, that T_0 has a Fourier series which is summable at $t = 0$ in the sense of Definition 2.12, with the sum $-1/12$.

As $T \mapsto \mathbf{H}_\lambda T : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is continuous, it follows from (12) that

$$\mathbf{H}_2 T_0 = e^{2it} + 2e^{4it} + 3e^{6it} + 4e^{8it} + 5e^{10it} + \cdots. \tag{13}$$

Moreover,

$$4\mathbf{H}_2 T_0 = 4e^{2it} + 8e^{4it} + 12e^{6it} + 16e^{8it} + 20e^{10it} + \cdots. \tag{14}$$

Subtracting (12) and (14), we obtain

$$
\begin{aligned}
T_0 - 4\mathbf{H}_2 T_0 &= e^{it} + 2e^{2it} + 3e^{3it} + 4e^{4it} + 5e^{5it} + 6e^{6it} + \cdots \\
&\quad - 4e^{2it} \qquad\quad - 8e^{4it} \qquad\quad - 12e^{6it} - \cdots \\
&= e^{it} - 2e^{2it} + 3e^{3it} - 4e^{4it} + 5e^{5it} - 6e^{6it} + - \cdots.
\end{aligned}
$$

Thus we arrive at the identity

$$T_0 - 4\mathbf{H}_2 T_0 = \operatorname{Pf} \frac{e^{it}}{(1 + e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta'_{(2n+1)\pi} =: S. \tag{15}$$

But we know from the previous section that $\sum_{n \in \mathbb{Z}} c_n(S) = \frac{1}{4}$.

Hence it follows from Definition 2.12 that

$$1 + 2 + 3 + \cdots = \sum_{n \in \mathbb{Z}} c_n(T_0) = \frac{1}{1 - 4} \sum_{n \in \mathbb{Z}} c_n(S) = \frac{1}{-3} \cdot \frac{1}{4} = -\frac{1}{12}.$$

Here we used the (extended) summability notion from Definition 2.12, and so we have shown that $T_0 \in \mathbf{\Sigma}$. Had T_0 belonged to $\boldsymbol{\sigma}$, we would of course obtain the same sum of the Fourier coefficients of T_0 at $t = 0$. But we now show (in Proposition 7.1 below) that in fact $T_0 \notin \boldsymbol{\sigma}$ (that is, the distribution T_0 has a Fourier series that is *not* summable at $t = 0$ *in the restricted sense of Definition 2.4*).

Proposition 7.1. $T_0 := \sum_{n=1}^{\infty} n e^{nit} \notin \boldsymbol{\sigma}$.

Proof. With $\mathbf{T}_\pi$ denoting the translation operator by π, which we note is a continuous map linear map from $\mathcal{D}'(\mathbb{R})$ to itself, we have

$$
\begin{aligned}
\mathbf{T}_\pi T_0 &= \mathbf{T}_\pi \sum_{n=1}^{\infty} n e^{nit} = \sum_{n=1}^{\infty} \mathbf{T}_\pi n e^{nit} \\
&= \sum_{n=1}^{\infty} n e^{ni(t-\pi)} = \sum_{n=1}^{\infty} (-1)^n n e^{nit} = -S,
\end{aligned}
$$

and so

$$T_0 = -\mathbf{T}_{-\pi}S. \tag{16}$$

Using this, we now calculate (for later use below) the action of T_0 on test functions of a special type. For any $\chi \in \mathcal{D}(\mathbb{R})$ with the properties that $\mathrm{supp}(\chi) \subset (-\pi/2, \pi/2)$ and $\chi(0) = \chi'(0) = 0$, we have

$$\langle T_0, \chi \rangle = -\langle \mathbf{T}_{-\pi}S, \chi \rangle = -\langle S, \mathbf{T}_\pi \chi \rangle$$

$$= -\lim_{\epsilon \searrow 0} \left(\int_{(0,\pi-\epsilon)\cup(\pi+\epsilon,2\pi)} \frac{e^{it}\chi(t-\pi)}{(1+e^{it})^2} dt - \frac{\chi(\pi-\pi)}{\tan(\epsilon/2)} \right)$$

$$= -\lim_{\epsilon \searrow 0} \left(-\int_{(-\pi,-\epsilon)\cup(\epsilon,\pi)} \frac{e^{i\tau}\chi(\tau)}{(1-e^{i\tau})^2} d\tau - \frac{0}{\tan(\epsilon/2)} \right)$$

$$= \lim_{\epsilon \searrow 0} \int_{(-\pi,-\epsilon)\cup(\epsilon,\pi)} \frac{e^{i\tau}\chi(\tau)}{(1-e^{i\tau})^2} d\tau$$

In the above, we used the substitution $\tau = t - \pi$ in order to obtain the equality in the third row.

Summarising, for $\chi \in \mathcal{D}(\mathbb{R})$ with the properties that $\mathrm{supp}(\chi) \subset (-\pi/2, \pi/2)$ and $\chi(0) = \chi'(0) = 0$, we have

$$\langle T_0, \chi \rangle = \lim_{\epsilon \searrow 0} \int_{(-\pi,-\epsilon)\cup(\epsilon,\pi)} \frac{e^{i\tau}\chi(\tau)}{(1-e^{i\tau})^2} d\tau. \tag{17}$$

This fact will be used below, where χ will be replaced by the elements of an approximate identity.

Let ψ be a nonzero test function in $\mathcal{D}(\mathbb{R})$ which is symmetric and nonnegative, and has its support $\mathrm{supp}(\psi) \subset (-1,1) \subset (-\pi/2, \pi/2)$. Define $\varphi \in \mathcal{D}(\mathbb{R})$ by

$$\varphi(t) = \frac{t^4 \psi(t)}{\displaystyle\int_{-1}^{1} t^4 \psi(t) dt} = \frac{1}{I} \cdot t^4 \psi(t) \quad (t \in \mathbb{R}), \text{ where } I := \int_{-1}^{1} t^4 \psi(t) dt > 0.$$

Then φ is a symmetric positive mollifier with $\varphi(0) = \varphi'(0) = 0$. Analogously, we also define $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R})$ by

$$\widetilde{\varphi}(t) = \frac{t^2 \psi(t)}{\displaystyle\int_{-1}^{1} t^2 \psi(t) dt} = \frac{1}{\widetilde{I}} \cdot t^2 \psi(t) \quad (t \in \mathbb{R}), \text{ where } \widetilde{I} := \int_{-1}^{1} t^2 \psi(t) dt > 0.$$

Then $\widetilde{\varphi}$ is a symmetric positive mollifier with $\widetilde{\varphi}(0) = \widetilde{\varphi}'(0) = 0$.

In (17), if in particular, we take $\chi = \varphi_m$, then we obtain

$$
\begin{aligned}
\langle T_0, \varphi_m \rangle &= \lim_{\epsilon \searrow 0} \int_{(-\pi,-\epsilon)\cup(\epsilon,\pi)} \frac{e^{it}\varphi_m(t)}{(1-e^{it})^2} dt \\
&= \frac{1}{I} \cdot \lim_{\epsilon \searrow 0} \int_{(-\pi,-\epsilon)\cup(\epsilon,\pi)} \frac{e^{it} m \cdot m^4 t^4 \psi(mt)}{(1-e^{it})^2} dt \\
&= m^2 \frac{\widetilde{I}}{I} \cdot \lim_{\epsilon \searrow 0} \int_{(-\pi,-\epsilon)\cup(\epsilon,\pi)} \frac{m \cdot m^2 t^2 \psi(mt)}{\widetilde{I}} \frac{t^2 e^{it}}{(1-e^{it})^2} dt \\
&= m^2 \frac{\widetilde{I}}{I} \cdot \int_{-\pi}^{\pi} \widetilde{\varphi}_m(t) \frac{t^2 e^{it} \beta(t)}{(1-e^{it})^2} dt,
\end{aligned}
$$

where $\beta \in \mathcal{D}(\mathbb{R})$ is such that $\beta \equiv 1$ on $(-\pi, \pi)$ and has compact support contained inside $(-2\pi, 2\pi)$. Hence

$$
\langle T_0, \varphi_m \rangle = m^2 \frac{\widetilde{I}}{I} \cdot \left\langle \widetilde{\varphi}_m, \frac{t^2 e^{it} \beta(t)}{(1-e^{it})^2} \right\rangle,
$$

where in right hand side of the equality, we view $\widetilde{\varphi}_m$ as the regular distribution in $\mathcal{E}'(\mathbb{R})$ corresponding to $\widetilde{\varphi}_m$, and

$$
t \mapsto \frac{t^2 e^{it} \beta(t)}{(1-e^{it})^2}
$$

as a test function in $\mathcal{E}(\mathbb{R})$. Then, since

$$
\lim_{m\to\infty} \widetilde{\varphi}_m = \delta_0
$$

in $\mathcal{D}'(\mathbb{R})$, it follows that

$$
\begin{aligned}
\lim_{m\to\infty} \left\langle \widetilde{\varphi}_m, \frac{t^2 e^{it} \beta(t)}{(1-e^{it})^2} \right\rangle &= \left\langle \delta_0, \frac{t^2 e^{it} \beta(t)}{(1-e^{it})^2} \right\rangle = \lim_{t\to 0} \frac{t^2 e^{it} \beta(t)}{(1-e^{it})^2} \\
&= \lim_{t\to 0} \frac{e^{it} \beta(t)}{i^2 \left(\dfrac{1-e^{it}}{-it} \right)^2} = \frac{e^{i0} \cdot \beta(0)}{i^2 \left(\dfrac{d}{dz} e^z \Big|_{z=0} \right)^2} \\
&= \frac{1 \cdot 1}{-1 \cdot 1^2} = -1.
\end{aligned}
$$

Consequently,

$$
\lim_{m\to\infty} \langle T_0, \varphi_m \rangle = \lim_{m\to\infty} m^2 \frac{\widetilde{I}}{I} \cdot (-1) = -\infty,
$$

showing that $T_0 \notin \boldsymbol{\sigma}$. $\square$

8. COMPUTATION OF $1^k + 2^k + 3^k + \cdots$ FOR $k \in \mathbb{N}$

We will first show the following result, where $S^{(k)}$ denotes the kth order derivative of the distribution S, where S is the distribution

$$S := \mathrm{Pf}\frac{e^{it}}{(1 + e^{it})^2} + i\pi \sum_{n\in\mathbb{Z}} \delta'_{(2n+1)\pi}.$$

It is easy to see, using the fact that the translation operator commutes with the differentiation operator for test functions, that $S^{(k)}$ is also 2π-periodic.

Proposition 8.1. *For all $k \in \mathbb{N}$, the 2π-periodic distribution $S^{(k)} \in \mathbf{\Sigma}$, that is, $S^{(k)}$ has a Fourier series that is summable at $t = 0$ in the sense of Definition 2.12.*

Proof. Let $(\varphi_m)_{m\in\mathbb{N}}$ be any approximate identity. For all large enough m, the support of φ_m is contained inside $(-\pi/2, \pi/2)$. Then we have

$$
\begin{aligned}
\lim_{m\to\infty} \langle S^{(k)}, \varphi_m \rangle &= \lim_{m\to\infty} (-1)^k \langle S, \varphi_m^{(k)} \rangle \\
&= \lim_{m\to\infty} (-1)^k \int_{-\pi/2}^{\pi/2} \frac{\varphi_m^{(k)}(t)e^{it}}{(1 + e^{it})^2}dt \\
&= \lim_{m\to\infty} (-1)^k \left\langle \varphi_m^{(k)}, \frac{e^{it}}{(1 + e^{it})^2} \right\rangle \\
&= \lim_{m\to\infty} \left\langle \varphi_m, \left(\frac{d}{dt}\right)^k \frac{e^{it}}{(1 + e^{it})^2} \right\rangle \\
&= \left\langle \delta_0, \left(\frac{d}{dt}\right)^k \frac{e^{it}}{(1 + e^{it})^2} \right\rangle \\
&= \left. \left(\frac{d}{dt}\right)^k \frac{e^{it}}{(1 + e^{it})^2} \right|_{t=0}.
\end{aligned}
\tag{18}
$$

This completes the proof. $\square$

We will also need the commutation relation between the operations of $\mathbf{H}_2$ and differentiation below.

Lemma 8.2. *For any distribution $T \in \mathcal{D}'(\mathbb{R})$, any $\lambda > 0$, and any $k \in \mathbb{N}$,*

$$(\mathbf{H}_\lambda T)^{(k)} = \lambda^k \mathbf{H}_\lambda \left(T^{(k)}\right).$$

Proof. For any test function $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$
\begin{aligned}
\left\langle (\mathbf{H}_\lambda T)^{(k)}, \varphi \right\rangle &= (-1)^k \left\langle \mathbf{H}_\lambda T, \varphi^{(k)} \right\rangle \\
&= \frac{(-1)^k}{\lambda} \left\langle T, \mathbf{H}_{\frac{1}{\lambda}} \left(\varphi^{(k)} \right) \right\rangle \\
&= \frac{(-1)^k}{\lambda} \left\langle T, \lambda^k \left(\mathbf{H}_{\frac{1}{\lambda}} \varphi \right)^{(k)} \right\rangle \\
&= \frac{\lambda^k}{\lambda} \left\langle T^{(k)}, \mathbf{H}_{\frac{1}{\lambda}} \varphi \right\rangle \\
&= \lambda^k \left\langle \mathbf{H}_\lambda \left(T^{(k)} \right), \varphi \right\rangle \\
&= \left\langle \lambda^k \cdot \mathbf{H}_\lambda \left(T^{(k)} \right), \varphi \right\rangle.
\end{aligned}
$$

Hence $(\mathbf{H}_\lambda T)^{(k)} = \lambda^k \mathbf{H}_\lambda \left(T^{(k)} \right)$. $\qquad\square$

As $T_0 - 4\mathbf{H}_2 T_0 = S$, we obtain by differentiating k times that

$$
T_0^{(k)} - 4 \cdot 2^k \cdot \mathbf{H}_2(T_0^{(k)}) = S^{(k)}.
$$

As $S^{(k)} \in \boldsymbol{\sigma}$, and since $k := 4 \cdot 2^k \neq 1$, we conclude that the Fourier series of $T_0^{(k)}$ is summable at $t = 0$, and we have

$$
\begin{aligned}
\sum_{n \in \mathbb{Z}} c_n(T_0^{(k)}) &= \frac{1}{1 - 4 \cdot 2^k} \sum_{n \in \mathbb{Z}} c_n(S^{(k)}) \\
&= \frac{1}{1 - 4 \cdot 2^k} \cdot \left(\frac{d}{dt} \right)^k \left. \frac{e^{it}}{(1 + e^{it})^2} \right|_{t=0}.
\end{aligned}
$$

But, because of the convergence of the Fourier series of T_0 in $\mathcal{D}'(\mathbb{R})$, we obtain by termwise differentiation of

$$
T_0 = e^{it} + 2e^{2it} + 3e^{3it} + \cdots
$$

that

$$
T_0^{(k-1)} = i^{k-1}(e^{it} + 2^k e^{2it} + 3^k e^{3it} + \cdots),
$$

and so we obtain

$$
\begin{aligned}
1^k + 2^k + 3^k + \cdots &= \frac{1}{i^{k-1}} \cdot \frac{1}{1 - 4 \cdot 2^{k-1}} \cdot \left(\frac{d}{dt} \right)^{k-1} \left. \frac{e^{it}}{(1 + e^{it})^2} \right|_{t=0} \\
&= \frac{1}{i^{k-1}} \cdot \frac{1}{1 - 2^{k+1}} \cdot \left(\frac{d}{dt} \right)^{k-1} \left. \frac{e^{it}}{(1 + e^{it})^2} \right|_{t=0}. \quad (19)
\end{aligned}
$$

We note that (19) gives, for example,

$$
\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots &= \frac{1}{i(1-4\cdot 2)}\frac{d}{dt}\frac{e^{it}}{(1+e^{it})^2}\Big|_{t=0} \\
&= -\frac{ie^{it}}{7i}\left(\frac{1}{(1+e^{it})^2} - \frac{2e^{it}}{(1+e^{it})^3}\right)\Big|_{t=0} \\
&= -\frac{i}{7i}\left(\frac{1}{4} - \frac{2}{8}\right) = 0.
\end{aligned}
$$

As one more example, (19) gives

$$
\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots &= \frac{1}{i^2(1-4\cdot 4)}\frac{d^2}{dt^2}\frac{e^{it}}{(1+e^{it})^2}\Big|_{t=0} \\
&= \frac{1}{15}\left(-\frac{e^{it}}{(1+e^{it})^2} + \frac{6(e^{it})^2}{(1+e^{it})^3} - \frac{6(e^{it})^3}{(1+e^{it})^4}\right)\Big|_{t=0} \\
&= \frac{1}{15}\left(\frac{1}{4} + \frac{6}{8} - \frac{6}{16}\right) = \frac{1}{120}.
\end{aligned}
$$

We note that these values match the values of the usual analytic continuation of the Riemann zeta function $\zeta(s)$ at $s = -2$ and at $s = -3$ respectively; see for example [13, §8.2]. Here ζ denotes the Riemann-zeta function defined by

$$
\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.
$$

The series for $\zeta(s)$ converges absolutely if $\mathrm{Re}(s) > 1$, but it diverges whenever $\mathrm{Re}(s) \leqslant 1$. However, one can use an analytic continuation of the zeta function in the punctured plane $\mathbb{C}\backslash\{1\}$ for determining the zeta function at points where the series fails to converge. The analytic continuation satisfies a functional equation (see for example [14, Theorem 2.1]), which yields for $k \in \mathbb{N}$ that

$$
\zeta(-k) = \frac{2}{(2\pi)^{k+1}}\sin\left(-k\frac{\pi}{2}\right)k!\zeta(1+k); \tag{20}
$$

We justify in the next two results below that our formula (19) coincides with $\zeta(-k)$ for all $k \in \mathbb{N}$. Proposition 8.3 is due to Arne Meurman (personal communication).

Proposition 8.3. *For all odd $k \in \mathbb{N}$,*

$$
\zeta(-k) = \frac{1}{i^{k-1}}\frac{1}{1-2^{k+1}}\left(\frac{d}{dz}\right)^{k-1}\frac{e^{iz}}{(1+e^{iz})^2}\Big|_{z=0}.
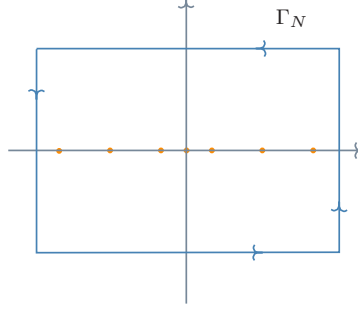$$

Proof. Let k be a positive odd integer and set

$$
f(z) = \frac{1}{z^{k+1}(1+e^{iz})}.
$$

We shall consider $\int_{\Gamma_N} f(z)dz$, where Γ_N denotes the contour (shown below)

$$[-N\pi(1+i), -N\pi i + N\pi, N\pi i + N\pi, N\pi i - N\pi, -N\pi(1+i)],$$

and N is a positive even integer that shall approach ∞.

Estimating, one find that $|f(z)| = O(N^{-k-1})$ on Γ_N, so that

$$\lim_{N\to\infty} \int_{\Gamma_N} f(z)dz = 0.$$

As f has poles at $z = 0$ and $z = (2j+1)\pi$, $j \in \mathbb{Z}$, the residue theorem gives

$$0 = \operatorname{Res}_{z=0} f(z) + \sum_{j\in\mathbb{Z}} \operatorname{Res}_{z=(2j+1)\pi} f(z). \tag{21}$$

One obtains

$$\operatorname{Res}_{z=(2j+1)\pi} f(z) = \frac{1}{[(2j+1)\pi]^{k+1}(-i)},$$

so that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \operatorname{Res}_{z=(2j+1)\pi} f(z) &= \frac{2i}{\pi^{k+1}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^{k+1}} \\
&= \frac{2i}{\pi^{k+1}} \sum_{j=0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{k+1}} - \sum_{n=1}^{\infty} \frac{1}{(2n)^{k+1}} \right) \\
&= \frac{2i}{\pi^{k+1}} \left(1 - \frac{1}{2^{k+1}} \right) \zeta(k+1).
\end{aligned}$$

Moreover,

$$\begin{aligned}
-\operatorname{Res}_{z=0} f(z) &= -\frac{1}{k!} \left(\frac{d}{dz}\right)^k \frac{1}{1+e^{iz}} \bigg|_{z=0} \\
&= -\frac{1}{k!} \left(\frac{d}{dz}\right)^{k-1} \frac{(-i)e^{iz}}{(1+e^{iz})^2} \bigg|_{z=0} \\
&= \frac{i}{k!} \left(\frac{d}{dz}\right)^{k-1} \frac{e^{iz}}{(1+e^{iz})^2} \bigg|_{z=0}.
\end{aligned}$$

Thus (21) gives

$$\frac{2i}{\pi^{k+1}}\left(1-\frac{1}{2^{k+1}}\right)\zeta(k+1) = \frac{i}{k!}\left(\frac{d}{dz}\right)^{k-1}\frac{e^{iz}}{(1+e^{iz})^2}\bigg|_{z=0}. \qquad (22)$$

Recall that the functional equation for ζ gives

$$\zeta(-k) = \frac{2(-1)^{\frac{k+1}{2}}k!}{(2\pi)^{k+1}}\zeta(k+1). \qquad (23)$$

Comparing (22) and (23) gives the formula

$$\zeta(-k) = \frac{1}{i^{k-1}}\frac{1}{1-2^{k+1}}\left(\frac{d}{dz}\right)^{k-1}\frac{e^{iz}}{(1+e^{iz})^2}\bigg|_{z=0}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 8.4. *For all even $k \in \mathbb{N}$,*

$$\zeta(-k) = 0 = \frac{1}{i^{k-1}}\frac{1}{1-2^{k+1}}\left(\frac{d}{dz}\right)^{k-1}\frac{e^{iz}}{(1+e^{iz})^2}\bigg|_{z=0}.$$

Proof. For even $k \in \mathbb{N}$, since $\sin\left(-k\frac{\pi}{2}\right) = 0$, it follows that $\zeta(-k) = 0$. We will show that in our formula (19), also

$$\left(\frac{d}{dt}\right)^{k-1}\frac{e^{it}}{(1+e^{it})^2}\bigg|_{t=0} = 0$$

for even k, so that (19) matches $\zeta(-k) = 0$. Set

$$g(z) = \frac{1}{1+e^z}.$$

Then g is holomorphic in a neighbourhood of 0. So $h := g(i\cdot)$ is also holomorphic in a neighbourhood of 0, and by a repeated application of the chain rule,

$$\left(\frac{d}{dz}\right)^k h\bigg|_{z=0} = i^k\left(\frac{d}{dz}\right)^k g\bigg|_{z=i0=0}.$$

But

$$\left(\frac{d}{dz}\right)^k h\bigg|_{z=0} = \left(\frac{d}{dz}\right)^{k-1}\frac{d}{dz}\frac{1}{1+e^{iz}}h\bigg|_{z=0} = \left(\frac{d}{dz}\right)^{k-1}\frac{-ie^{iz}}{(1+e^{iz})^2}\bigg|_{z=0}$$

$$= -i\left(\frac{d}{dz}\right)^{k-1}\frac{e^{iz}}{(1+e^{iz})^2}\bigg|_{z=0},$$

and so

$$\left(\frac{d}{dz}\right)^{k-1}\frac{e^{iz}}{(1+e^{iz})^2}\bigg|_{z=0} = i\left(\frac{d}{dz}\right)^k h\bigg|_{z=0} = ii^k\left(\frac{d}{dz}\right)^k g\bigg|_{z=0}. \qquad (24)$$

We show that the right most expression in (24) is 0. We have

$$g(z) = \frac{1}{1+e^z} = \frac{1}{2} - \frac{1}{2}\frac{e^{z/2} - e^{-z/2}}{e^{z/2} + e^{-z/2}} = \frac{1}{2} - \frac{1}{2}\tanh\frac{z}{2}.$$

For $k \geqslant 2$,

$$\left(\frac{d}{dz}\right)^k g = \left(\frac{d}{dz}\right)^k \left(\frac{1}{2} - \frac{1}{2}\tanh\frac{z}{2}\right) = 0 - \frac{1}{2}\left(\frac{d}{dz}\right)^k \tanh\frac{z}{2}.$$

But as $z \mapsto \tanh\frac{z}{2}$ is an odd function, it follows that for even k,

$$z \mapsto \left(\frac{d}{dz}\right)^k \tanh\frac{z}{2}$$

is an odd function too, and in particular, it vanishes at $z = 0$. Hence $\left(\frac{d}{dz}\right)^k g\Big|_{z=0} = 0$, and using (24), also

$$\left(\frac{d}{dt}\right)^{k-1} \frac{e^{it}}{(1 + e^{it})^2}\Bigg|_{t=0} = 0.$$

This completes the proof.   □

9. APPENDIX A: CASIMIR EFFECT

In quantum field theory, upon quantising a classical field, one ends up with infinitely many harmonic oscillators, one at each spacetime point. If we take this picture seriously, then we run into the problem of having to add up all of their ground state energies, and taking that as the ground state energy of the quantum field. The Casimir effect, predicted in 1948 [2], allows the experimental demonstration [4] of the existence of this ground state energy obtained by summing the ground state energies of all the oscillators.
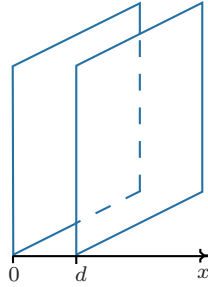
Consider two parallel uncharged large plane conductors of area A separated by a small distance d in empty space, as shown. This is like a capacitor, but the plates do not have any charge on them. From the classical point of view, there should not be any electromagnetic force between them. Nevertheless, it can be demonstrated experimentally that the two plates attract each other. This can be explained as follows. In the absence of the plates, the quantum electromagnetic field (a superposition of an infinite number of oscillators) is

in its ground state. But upon the introduction of the two plates, we impose perfect conductor boundary conditions for the electromagnetic field exactly at $x = 0$ and $x = d$. So the ground state energy for the oscillators will now be different (but still divergent) from the original ground state energy. The difference between the ground state energies, calculated with and without the plates, will involve subtracting one infinity from another, but seems to be formally finite as we will see in the toy example below, and can be used to predict the correct magnitude of the Casimir force, which has been experimentally verified [4].

As a simplified toy model, consider a massless scalar field φ with only one space dimension, and suppose that the field is constrained to vanish at $x = 0$ and $x = d$. The allowed oscillator modes

$$\varphi_n(x) = c_n \sin \frac{n\pi x}{d}, \quad n = 0, 1, 2, 3, \cdots$$

have wave numbers given by

$$k_n = \frac{n\pi}{d}.$$

The corresponding travelling wave is then

$$\varphi_n(x, t) = c_n \sin(k_n x - \omega_n t), \quad x, t \in \mathbb{R},$$

with the speed of propagation

$$c = \frac{\omega_n}{k_n}.$$

The associated ground state energy with a quantum harmonic oscillator with angular frequency ω is

$$E = \frac{\hbar \omega}{2}.$$

Thus we now obtain that the total ground state energy in the superposition of all possible modes is

$$
\begin{aligned}
E(d) &= \sum_{n=1}^{\infty} \frac{\hbar \omega_n}{2} \\
&= \frac{\pi \hbar c}{2d} \sum_{n=1}^{\infty} n \\
&= -\frac{\pi c \hbar}{24d}.
\end{aligned}
$$

If we imagine two point particles at $x = 0$ and at $x = d$ (analogous to the two conducting plates), then this energy leads to an attractive force of magnitude

$$E'(d) = \frac{\pi c \hbar}{24d^2}.$$

10. APPENDIX B: ZETA FUNCTION REGULARISATION

In order for easy comparison and contrast of our summation method versus the zeta regularisation method, both used in showing

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}, \tag{25}$$

we outline briefly here the idea behind the zeta function regularisation method.

One first considers the Riemann zeta function, given by

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

So we would like to set $s = -1$, in order to get the desired sum

$$1 + 2 + 3 + \cdots.$$

The series for $\zeta(s)$ converges absolutely if $\text{Re}(s) > 1$, but it diverges whenever $\text{Re}(s) \leqslant 1$. However, one can use an analytic continuation of the zeta function in the punctured plane $\mathbb{C}\backslash\{1\}$ for determining the zeta function at points where the series fails to converge. In [12], it was shown that a series derived using Euler's transformation provides the analytic continuation of ζ for all complex numbers $s \neq 1$, and in particular at negative integers, the series becomes a finite sum, whose value is given by an explicit formula for Bernoulli numbers. In particular, this formula then yields $\zeta(-1) = -1/12$. Our summation method yields

$$1^k + 2^k + 3^k + \cdots = \frac{1}{i^{k-1}} \cdot \frac{1}{1 - 2^{k+1}} \cdot \left(\frac{d}{dt}\right)^{k-1} \frac{e^{it}}{(1 + e^{it})^2}\bigg|_{t=0}. \tag{26}$$

We had seen in Proposition 8.3 of Section 8, our formula (26) gives values matching exactly the corresponding values of $\zeta(-k)$.

However, our route to arriving at $1^k + 2^k + 3^k + \cdots$ is quite different from the analytic continuation of the zeta function, since we rely on distribution theory. The formula (26) we obtained is afforded in particular by $(*)$, $(\star)$ on page 19, and (18) on page 23, where the numbers

$$\left(\frac{d}{dt}\right)^k \frac{e^{it}}{(1 + e^{it})^2}\bigg|_{t=0}$$

appear.

The Fourier series appears in quantum field theory computations, and hence it is conceivable that our summation method is more natural in this context. We refer the reader to [7], where the Casimir effect is analysed using the framework of quantum field theory, and in particular to [7, §III], where Fourier series plays a role.

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