Loss Sharing: Characterizing a New Class of Rules[1]

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Abstract
The class of rules that we propose and characterize can be viewed as a variant of the standard model in the literature on cost and surplus sharing. It basically has two reference points: an equal share of the loss and a weighted difference between an agent's endowment or claim and the average endowment of the individuals concerned. Our class of rules comprises some prominent sharing rules such as equal split and the proportionality principle.

Keywords: loss sharing; sharing rule; monotonicity in contributions; independence of rank-and-mean preserving changes

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1. Introduction

The object of our study are rules that propose solutions in situations in which losses have to be shared. We consider cases where several agents who possess some initial endowment face

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a situation where they are asked, perhaps forced, to give something away so that they will no longer be able to keep their status quo allocation, and we are looking for rules which may be acceptable to the persons concerned to resolve such situations. Cases such as these have been considered from different angles in economics. Closest to what we have in mind, though still substantially different, are situations of bankruptcy and bequest. In the former, various agents have claims of differing size against a bank or firm and the liquidation value that remains is not sufficient to satisfy all entitlements. In the latter, a father, let's say, made promises to his children but after his death, the heirs find out that the estate he left behind is not large enough to honor all promises. Another case, and an important one, is taxation which in the words of Young (1988, p. 322) perhaps is "the most familiar example of a distribution problem involving loss".

Various theoretical concepts exist in the literature which prescribe how a deficit or loss should be shared. Most prominent are the proportional solution and the egalitarian rule (Moulin, 1987, 2002) but also the constrained equal-awards solution and the constrained equal-losses rule (Herrero and Villar, 2001), not to forget the Talmud rule (Aumann and Maschler, 1985), the latter being a compromise between the constrained equal-awards and the constrained equal-losses rule. Young (1988) provided a characterization of equal sacrifice methods in taxation. One should also mention contributions by Shapley (1953), O'Neil (1982), Pfingsten (1991), and the comprehensive investigations by Thomson (2003, 2013, 2015, 2019) on the adjudication of conflicting claims and the division problem in the light of resource allocation. The latter aspect has also recently been addressed by Ju and Moreno-Ternero (2018). Ju et al. (2007) characterize non-manipulable division rules which may treat an agent differently from other persons based on this agent's characteristic vector. Finally, Hougaard et al. (2012) generalize the analysis of Thomson and Yeh (2008) who developed the concept of operators on the space of rules. Hougaard et al. introduce the notion of baselines which represent some reference point in the division problem of adjudicating conflicting claims.

The class of rules that we propose and characterize in this paper can be viewed as another variant of the standard model in the literature on cost and surplus sharing. It basically has two reference points: an equal share of the loss and a weighted difference between an agent's endowment or claim and the average endowment of the people concerned. The following section presents the model and provides its axiomatic characterization. A few final remarks are offered in section 3.

2. The model

Let $N = \{1, \cdots, n\}$ denote the set of individuals in society with $n \geq 2$. Each individual $i \in N$ is endowed with a certain amount of resources, money let's say, $\omega_i \geq 0$. The society incurs a loss L with $0 < L \leq \sum_{i \in N} \omega_i$. $(L, (\omega_1, \cdots, \omega_n))$ is called a loss division problem. L is given and fixed. From now on, we shall simply refer to $(\omega_1, \cdots, \omega_n)$ as a loss division problem.

Let Ξ denote the following set of problems:

$$\{(\omega_1, \cdots, \omega_n): (0 < L \leq \textstyle\sum_{i \in N} \omega_i),\ (\omega_i \geq 0\ \ \forall i \in N)\}.$$

For any $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$, let

$$\mu(\xi) = \frac{\omega_1 + \cdots + \omega_n}{n}$$

denote the mean of $\omega_1, \cdots, \omega_n$. For any $\bar{\mu} > 0$, let $\Xi(\bar{\mu})$ denote the set

$$\{\xi \in \Xi : \mu(\xi) = \bar{\mu}\}.$$

Let $\mathbf{R}_+$ be the set of all non-negative numbers and $\mathbf{R}_+^n$ as the n-fold Cartesian product of $\mathbf{R}_+$. A sharing rule $f: \Xi \to \mathbf{R}_+^n$ is such that, for each $\xi \in \Xi$, $[f_i(\xi) \in \mathbf{R}_+$ for all $i \in N]$ and $[\sum_{i \in N} f_i(\xi) \geq L]$, where $f_i(\xi)$ is the loss that individual i incurs.

We now consider the class of rules that we wish to discuss and characterize in this paper.

A sharing rule f is the $(\frac{L}{n}, \mu)$-referenced rule if, for each $\bar{\mu} > 0$, there exists $\alpha \in \mathbf{R}_+$ such that

$$\forall \xi \in \Xi(\bar{\mu}), \quad \forall i \in N: f_i(\xi) = \frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big).$$

Therefore, the $(\frac{L}{n}, \mu)$-referenced rule uses the average "burden" to everyone, $\frac{L}{n}$, as reference and then adds to this reference level a proportion of the gap between an individual's endowment and the mean of individuals' endowments to compute individual i's share. The second term can be viewed as a position-dependent corrective to an equal division of the overall loss.

Remark 1. *We note that the $(\frac{L}{n}, \mu)$-referenced rule establishes a class of rules that depend on the choice of α, some prominent members of the class are given below. Note also that α may depend on the mean of a problem.*

Remark 2. *If $\alpha = 0$ or if there is equality in initial endowments, the $(\frac{L}{n}, \mu)$-referenced rule becomes the equal-loss-division rule.*

Remark 3. *If $\alpha = 1$, the $(\frac{L}{n}, \mu)$-referenced rule leads to an equalization of final endowments of all agents.*

Remark 4. *If $\alpha = \frac{L}{n\mu}$, then $f_i(\xi) = \alpha\omega_i$ for all $i \in N$, and, as a consequence, the $(\frac{L}{n}, \mu)$-referenced rule becomes the proportional rule.*

Remark 5. *When $\alpha = \frac{L}{c + \sum_{i \in N} \omega_i}$, where c is a positive constant, then* $\forall \xi = (\omega_1, \cdots, \omega_n) \in \Xi(\bar{\mu}), \quad \forall i \in N$:

$$f_i(\xi) = \frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big) = \frac{L\omega_i}{c + \sum_{j \in N} \omega_j} + \frac{L}{n}\left[1 - \frac{\sum_{j \in N} \omega_j}{c + \sum_{j \in N} \omega_j}\right]$$

The $(\frac{L}{n}, \mu)$-referenced rule becomes the rule that was proposed and axiomatically characterized by Pfingsten (1991).

The axioms that characterize our class of rules are “symmetry”, “efficiency”, “independence of rank-and-mean-preserving changes”, and “monotonicity in contributions”. They are defined and explained below.

Symmetry: For all $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$, if $\omega_1 = \cdots = \omega_n$, then $f_1(\xi) = \cdots = f_n(\xi)$.

Efficiency: For all $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$, $\sum_{i \in N} f_i(\xi) = L$.

Independence of Rank-and-Mean-Preserving Changes: For all $\xi = (\omega_1, \cdots, \omega_n)$ and $\xi' = (\omega'_1, \cdots, \omega'_n) \in \Xi$ with [for all $i, j \in N$, $\omega_i \leq \omega_j \Rightarrow \omega'_i \leq \omega'_j$], if $\mu(\xi) = \mu(\xi')$, then, for any $i \in N$, $[\omega_i = \omega'_i \Rightarrow f_i(\xi) = f_i(\xi')]$.

Monotonicity in Contributions: For all $\varepsilon \in \mathbf{R}_+$ with $\varepsilon \neq 0$, there exists a continuous function $g: \mathbf{R}_+ \to \mathbf{R}_+$ such that, for all $i \in N$, all $\xi = (\omega_1, \cdots, \omega_n)$ and $\xi' = (\omega'_1, \cdots, \omega'_n) \in \Xi$, if $\mu(\xi) = \mu(\xi')$ and $\omega'_i = \omega_i + \varepsilon$, then $f_i(\xi') = f_i(\xi) + g(\varepsilon)$.

The first two axioms are standard in distributional analyses. “Symmetry” says that if everyone has the same endowment, the total loss should be shared equally. Symmetry is a much weaker requirement than Anonymity and Equal Treatment of Equals which are commonly used in the related literature (see, for example, Moulin (2002), Ju et al. (2007), Hougaard et al. (2012)). “Efficiency” merely requires that the total contributions by all players must come up to the loss that is to be shared, also a standard requirement in the related literature.

To the best of our knowledge, the next axiom is new and does not have a counterpart in the literature on cost or surplus sharing to which we referred above.

“Independence of Rank-and-Mean-Preserving Changes” considers two sets of endowments with the same mean and the same rank order of the agents in terms of their endowments. This axiom postulates that if an individual has the same endowment in two situations, this individual’s share of total loss should be the same. This requirement is very natural since the individual’s position in the “hierarchy” of endowments has not changed. What this axiom, however, ignores are endowment differences as long as the hierarchy of endowments among the agents is preserved.

Finally, the axiom “Monotonicity in Contributions” considers incremental changes in the vector of initial endowments while the average endowment across all agents remains the same. It requires that if the endowment of a particular individual increases (decreases) from one situation to another, the contribution to the overall loss of this person should increase (decrease) as well. The attribute “monotonicity” refers to the demand that an agent’s change in initial endowment and this person’s share of the loss move in the same direction. This directional requirement is similar to an axiom called “Claim Monotonocity”, used by Ju et al. (2007) when those authors characterize a rule which is a convex combination of the proportional rule and equal division. As a matter of fact, the property of claim monotonicity can be found in Dagan et al. (1997) in connection with bankruptcy rules, in Yeh (2006) in relation to the constrained equal-awards solution, in Thomson’s extensive survey from 2003, and at other places. Finally, an axiom called aggregate monotonicity which goes back to

Megiddo (1974) has very recently been used by Calleja et al. (2020) within the setting of cooperative transferable utility games.

Theorem. *A sharing rule f on Ξ satisfies Symmetry, Efficiency, Independence of Rank-and-Mean-Preserving Changes, and Monotonicity in Contributions if and only if it is the $(\frac{L}{n},\mu)$-referenced rule.*

Proof. It can be checked that the $(\frac{L}{n},\mu)$-referenced rule satisfies Symmetry, Efficiency, Independence of Rank-and-Mean-Preserving Changes, and Monotonicity in Contributions. In what follows, we shall show that, if a sharing rule f on Ξ satisfies Symmetry, Efficiency, Independence of Rank-and-Mean-Preserving Changes, and Monotonicity in Contributions, then it is the $(\frac{L}{n},\mu)$-referenced rule.

Let f on Ξ satisfy the four conditions. Furthermore, let $\xi = (\omega_1,\cdots,\omega_n) \in \Xi$ and consider $\xi^\mu = (\omega_1^\mu,\cdots,\omega_n^\mu) \in \Xi$ such that $[\omega_i^\mu = \mu(\xi)$ for all $i \in N]$. By Symmetry and Efficiency,

$$f_i(\xi^\mu) = \frac{L}{n} \quad \text{for all} \quad i \in N.$$

Consider $\max\{\omega_1,\cdots,\omega_n\}$ and $\min\{\omega_1,\cdots,\omega_n\}$. If $\max\{\omega_1,\cdots,\omega_n\} = \min\{\omega_1,\cdots,\omega_n\}$, then $\omega_i = \mu(\xi)$ for all i ∈ N. It then follows immediately that, in this case, for all $i \in N$

$$f_i(\xi) = \frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big) \text{ for any } \alpha \geq 0.$$

Suppose now that $\max\{\omega_1,\cdots,\omega_n\} > \min\{\omega_1,\cdots,\omega_n\}$. Without loss of generality, let $\omega_n = \max\{\omega_1,\cdots,\omega_n\}$ and $\omega_1 = \min\{\omega_1,\cdots,\omega_n\}$. Let $\varepsilon = \min\{\omega_n - \mu(\xi), \mu(\xi) - \omega_1\}$. Note that $\varepsilon > 0$.

Consider the case in which $\varepsilon = \omega_n - \mu(\xi)$ first. Take any $\varepsilon_1, \varepsilon_2 \in [0,\varepsilon]$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and three vectors $\xi^1 = (\omega_1^1,\cdots,\omega_n^1), \xi^2 = (\omega_1^2,\cdots,\omega_n^2), \xi^3 = (\omega_1^3,\cdots,\omega_n^3) \in \Xi$ such that

$$\omega_1^1 = \mu(\xi) - \varepsilon_1, \quad \omega_n^1 = \mu(\xi) + \varepsilon_1, \quad \omega_j^1 = \mu(\xi) \quad \forall j \in N\backslash\{1,n\}$$

$$\omega_1^2 = \mu(\xi) - \varepsilon_2, \quad \omega_n^2 = \mu(\xi) + \varepsilon_2, \quad \omega_j^2 = \mu(\xi) \quad \forall j \in N\backslash\{1,n\}$$

$$\omega_1^3 = \mu(\xi) - \varepsilon, \quad \omega_n^3 = \mu(\xi) + \varepsilon, \quad \omega_j^3 = \mu(\xi) \quad \forall j \in N\backslash\{1,n\}.$$

By Monotonicity in Contributions, there exists a continuous function $g: \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$f_n(\xi^1) = f_n(\xi^\mu) + g(\varepsilon_1), \quad f_n(\xi^2) = f_n(\xi^\mu) + g(\varepsilon_2) \tag{1}$$

and

$$f_n(\xi^3) = f_n(\xi^\mu) + g(\varepsilon_1 + \varepsilon_2), \quad f_n(\xi^3) = f_n(\xi^1) + g(\varepsilon_2). \tag{2}$$

From equation (2), we have

$$f_n(\xi^\mu) + g(\varepsilon_1 + \varepsilon_2) = f_n(\xi^1) + g(\varepsilon_2).$$

Noting equation (1), we then obtain

$$g(\varepsilon_1+\varepsilon_2)=g(\varepsilon_1)+g(\varepsilon_2). \tag{3}$$

This is the Cauchy equation. Noting that $g(\cdot)\geq 0$, its unique solution is given by (see Aczél, 1966):

$$\text{for all } t \geqslant 0,\ g(t)=\alpha t \quad \text{for some constant } \alpha\in\mathbf{R}_+.$$

Therefore,

$$f_n(\xi^3)=f_n(\xi^\mu)+\alpha\big(\omega_n^3-\mu(\xi)\big)=\tfrac{L}{n}+\alpha\big(\omega_n^3-\mu(\xi)\big).$$

By Independence of Rank-and-Mean-Preserving Changes and noting that $\omega_n^3=\mu(\xi)+\varepsilon=\omega_n$ in this case, we have

$$f_n(\xi)=f_n(\xi^3)=\tfrac{L}{n}+\alpha\big(\omega_n-\mu(\xi)\big). \tag{4}$$

By Independence of Rank-and-Mean-Preserving Changes and noting that $\mu(\xi)=\mu(\xi^3)$ and $\omega_i^3=\mu(\xi)$ for all $i\in N\backslash\{1,n\}$, we also have

$$\forall i\in N\backslash\{1,n\}\colon\ f_i(\xi^\mu)=f_i(\xi^3)=\tfrac{\mathrm{L}}{\mathrm{n}}.$$

Applying the above arguments to f_1 and noting that, for the case analyzed here, $\varepsilon=\omega_n-\mu(\xi)=\min\{\omega_n-\mu(\xi),\mu(\xi)-\omega_1\}$, we can obtain

$$f_1(\xi^3)=f_1(\xi^\mu)-\alpha\big(\omega_n-\mu(\xi)\big)=\ \mathrm{L/n}-\alpha\big(\omega_n-\mu(\xi)\big).$$

Now, consider $\max\{\omega_1,\cdots,\omega_{n-1}\}$. Without loss of generality, let $\omega_{n-1}=\max\{\omega_1,\cdots,\omega_{n-1}\}$. By repeating the procedures for f_1 and f_n above, we can show that

$$f_1(\xi)=L/n+\alpha(\omega_1-\mu(\xi)),\ \text{ if } \min\{\omega_{n-1}-\mu(\xi),\mu(\xi)-\omega_1^3\}=\mu(\xi)-\omega_1^3$$

or

$$f_{n-1}(\xi)=L/n+\alpha(\omega_{n-1}-\mu(\xi)),\ \text{if } \min\{\omega_{n-1}-\mu(\xi),\mu(\xi)-\omega_1^3\}=\omega_{n-1}-\mu(\xi).$$

And by repeating the above, we can obtain

$$\text{for all }\ i\in N\colon f_i(\xi)=\tfrac{L}{n}+\alpha\big(\omega_i-\mu(\xi)\big). \tag{5}$$

The case in which $\varepsilon=\mu(\xi)-\omega_1$ can be dealt with analogously to the above argument, and we can obtain equation (5) as well. This completes the proof of the theorem. ■

Remark 6. It may be noted that the Theorem does not explicitly put further restrictions on the value of α that figures in the $(\frac{L}{n},\mu)$-referenced rule in which, $\forall\xi\in\Xi(\bar{\mu}),\ \forall i\in N\colon f_i(\xi)=$

$\frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big)$, though in the proof of the Theorem, we implicitly incorporated the following two properties of a sharing rule:

Nonnegative Contributions. For all $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$ and all $i \in N$, $f_{\mathrm{i}}(\xi) \geq 0$.

Nonnegative Wealth after Contributions. For all $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$ and all $i \in N$, $\omega_i - f_{\mathrm{i}}(\xi) \geq 0$.

In addition, we now introduce the following progressivity condition on contributions:

Progressivity. For all $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$ and all $i, j \in N$, if $\omega_i \geq \omega_j$, then $[(\omega_i - f_{\mathrm{i}}(\xi)) - (\omega_j - f_{\mathrm{j}}(\xi)) \geq 0, f_{\mathrm{i}}(\xi) - f_{\mathrm{j}}(\xi) \geq 0]$.

If we require a sharing rule to satisfy Nonnegative Contributions, Nonnegative Wealth after Contributions and Progressivity as stated above, then the α value that figures in the $(\frac{L}{n}, \mu)$-referenced rule, namely $\forall \xi \in \Xi(\bar{\mu}),\ \forall i \in N\colon f_i(\xi) = \frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big)$, can be further restricted as follows: if $\omega_i \neq \mu(\xi)$ for some $i \in N$, then

$$\max\{0, \frac{\frac{L}{n} - \min\{\omega_i : i \in N\}}{\mu(\xi) - \min\{\omega_i : i \in N\}}\} \leq \alpha \leq \min\{1, \frac{\frac{L}{n}}{\mu(\xi) - \min\{\omega_i : i \in N\}}\}$$

Note that $\forall \xi \in \Xi(\bar{\mu}),\ \forall i \in N\colon f_i(\xi) = \frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big)$. Then, noting that $f_i(\xi) - f_j(\xi) = \frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big) - \left(\frac{L}{n} + \alpha\left(\omega_j - \mu(\xi)\right)\right) = \alpha\big(\omega_i - \omega_j\big)$ and from Progressivity, it follows easily that $\alpha \geq 0$. Similarly, observing that $[\big(\omega_i - f_{\mathrm{i}}(\xi)\big) - \left(\omega_j - f_j(\xi)\right) = [(\omega_i - \left(\frac{L}{n} + \alpha\big(\omega_i - \mu(\xi)\big)\right) - \left(\omega_j - \left(\frac{L}{n} + \alpha\left(\omega_j - \mu(\xi)\right)\right)\right) = (1 - \alpha)\big(\omega_i - \omega_j\big)$ and again from Progressivity, it follows that $\alpha \leq 1$. Next, without loss of generality, let $\xi = (\omega_1, \cdots, \omega_n) \in \Xi$ be such that $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ and $\omega_1 < \mu(\xi)$. Note that $\min\{\omega_i : i \in N\} = \omega_1$. Nonnegative Contributions implies that $f_1(\xi) = \frac{L}{n} + \alpha\big(\omega_1 - \mu(\xi)\big) \geq 0$. Then, $\alpha \leq \frac{\frac{L}{n}}{\mu(\xi) - \omega_1}$ follows immediately. On the other hand, by Nonnegative Wealth after Contributions, $\omega_1 - \frac{L}{n} - \alpha\big(\omega_1 - \mu(\xi)\big) \geq 0$, which implies that $\alpha \geq \frac{\frac{L}{n} - \min\{\omega_i : i \in N\}}{\mu(\xi) - \min\{\omega_i : i \in N\}}$. Note that $\alpha \geq 0$ and $\alpha \leq 1$ as well. Therefore, $\alpha \geq \max\{0, \frac{\frac{L}{n} - \min\{\omega_i : i \in N\}}{\mu(\xi) - \min\{\omega_i : i \in N\}}\}$ and $\alpha \leq \min\{1, \frac{\frac{L}{n}}{\mu(\xi) - \min\{\omega_i : i \in N\}}\}$.

Remark 7. The logical independence of the axioms that figure in the Theorem can be checked. First, it may be noted that, when $n = 2$, the axiom of Independence of Rank-and-Mean Preserving Changes is redundant since it is vacuously satisfied. For $n \geq 3$, we consider the following sharing rules: $\forall \xi \in \Xi(\bar{\mu})$, (7.1) $f_1(\xi) = \frac{L}{2} + \alpha(\omega_1 - \mu(\xi))$, and $\forall i \in N\backslash\{1\}$: $f_i(\xi) = \frac{L/2}{n-1} + \alpha(\omega_i - \mu(\xi))$,

(7.2) $\forall i \in N: f_i(\xi) = \frac{L}{n} + \frac{\omega_i}{n\mu(\xi)}$,

(7.3) $\forall i \in N: f_i(\xi) = \frac{L}{n} + \frac{\alpha(\omega_i - \mu(\xi))}{\max\{\omega_i : i \in N\}}$,

(7.4) $\forall i \in N: f_i(\xi) = \frac{L}{n} - \frac{\omega_i}{n\mu(\xi)} + \frac{1}{n}$.

It can be verified that the sharing rule defined in (7.1) violates Symmetry while it satisfies the other three axioms; (7.2) violates Efficiency while satisfying the other three axioms. Rule (7.3) violates Independence of Rank-and-Mean Preserving Changes while it satisfies the other three axioms, and (7.4) violates Monotonicity in Contributions while satisfying the other three axioms.

Remark 8. It may be noted that the $(\frac{L}{n}, \mu)$-referenced rule, $\forall \xi \in \Xi(\bar{\mu})$, $\forall i \in N: f_i(\xi) = \frac{L}{n} + \alpha(\omega_i - \mu(\xi))$, gives a class of sharing rules. Do we have a `procedure' that can single out some specific members of this family of rules? In an interesting paper, Hougaard et al. (2012) study `operators' that map rules to each other (see also Thomson and Yeh (2008)) and uncover interesting structural properties of some well-known rules in the literature. Their study can be fruitfully applied in our context to identify certain members of the class of $(\frac{L}{n}, \mu)$-referenced rules. For example, for a loss division problem $(L, (\omega_1, \cdots, \omega_n))$, we define its *dual problem* as $(\sum_{i=1}^{n} \omega_i - L, (\omega_1, \cdots, \omega_n))$. For a given loss division problem $(L, (\omega_1, \cdots, \omega_n))$ and a $(\frac{L}{n}, \mu)$-referenced rule f, $\forall i \in N: f_i(\xi) = \frac{L}{n} + \alpha(\omega_i - \mu(\xi))$, we define the *dual rule* of f, to be denoted by f^d, as a $(\frac{L}{n}, \mu)$-referenced rule applied to the dual problem of the given loss division problem: $f^d = \frac{\sum_{i=1}^{n} \omega_i - L}{n} + \alpha(\omega_i - \mu(\xi))$. We can then impose a structural requirement on a member of the $(\frac{L}{n}, \mu)$-referenced rules: *a member rule must be self-dual*, i.e., $\omega - f = f^d$ (see Hougaard et al. (2012), p. 108). It can be checked that the proportional rule and the Talmud rule are self-dual, and if the parameter α is further assumed to be independent of the loss, then $\alpha = \frac{1}{2}$: given a loss division problem $(L, (\omega_1, \cdots, \omega_n))$, for each $i \in N$, $f_i(\xi) = \frac{L}{n} + \alpha(\omega_i - \mu(\xi))$ and $f_i^d(\xi) = \frac{\sum_{i=1}^{n} \omega_i - L}{n} + \alpha(\omega_i - \mu(\xi))$. Since f is self-dual, we have $\omega_i - \left[\frac{L}{n} + \alpha(\omega_i - \mu(\xi))\right] = \frac{\sum_{i=1}^{n} \omega_i - L}{n} + \alpha(\omega_i - \mu(\xi))$ for all $i \in N$; it can then be checked that $\alpha = \frac{1}{2}$. One could use other operators studied in Hougaard et al. (2012) to single out specific members of the family of the $(\frac{L}{n}, \mu)$-referenced rules and we leave this investigation to another occasion.

3. Final Remarks

We pointed out in the previous section that our class of rules comprises some prominent sharing rules such as equal split and the proportionality principle. However, it does not explicitly propose a non-trivial convex combination of the proportional rule and equal division as it is put forward in Ju et al. (2007). Depending on the value of the behavioral parameter alpha in combination with the endowment vector, one can obtain a situation in which the person with the lowest initial endowment is completely exempted from any burden sharing or, for a different alpha value, a case in which the agent "at the bottom" has to contribute only very little. Since our model satisfies continuity, this contribution can, of course, be made successively larger. We should mention that during recent years we ran laboratory experiments at different universities. We found that the proportionality principle did not get too much support among agents who were bargaining over the share of a loss in groups of four (Gaertner et al., 2019). As a final solution to which every group member had to agree, proportionality actually rarely occurred. On the other hand, the variant of our model that exempts the lowest endowed person received considerable backing among the participants (Gaertner et al., 2020). A strong contestant in our series of experiments was the constrained equal awards rule, well known from bankruptcy problems, which in our experimental set-up exempts the lowest two agents from any loss contribution. Such a loss division is, however, not compatible with our class of rules.

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