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Abstract:	We find the exact value of the Ramsey number $R(C_{2\ell},K_{1,n})$, when ℓ and $n=O(\ell^{10/9})$ are large. Our result is closely related to the behaviour of Turán number $\ex(N, C_{2\ell})$ for an even cycle whose length grows quickly with N.

THE RAMSEY NUMBER OF A LONG EVEN CYCLE VERSUS A STAR

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ABSTRACT. We find the exact value of the Ramsey number $R(C_{2\ell}, K_{1,n})$, when ℓ and $n = O(\ell^{10/9})$ are large. Our result is closely related to the behaviour of Turán number $\mathrm{ex}(N, C_{2\ell})$ for an even cycle whose length grows quickly with N.

1. INTRODUCTION

For a graph H by

$$\mathrm{ex}(N,H) = \max\{|E| : G = (V,E) \not\supseteq H \text{ and } |V| = N\}$$

we denote its Turán number. Let us recall that for graphs H with chromatic number at least three the asymptotic value of $\mathrm{ex}(N,H)$ was determined over fifty years ago by Erdős and Stone [8], and Erdős and Simonovits [7], while for most bipartite graphs H the behaviour of $\mathrm{ex}(N,H)$ is not well understood. Let us recall some results on the case when H is an even cycle $C_{2\ell}$. The best upper bound for $\mathrm{ex}(N, C_{2\ell})$ for general ℓ is due to Bukh and Jiang [4], who improved the classical theorem of Bondy and Simonovits [3] to

$$\mathrm{ex}(N, C_{2\ell}) \le 80\sqrt{\ell}\ln \ell N^{1+1/\ell} + 10\ell^2 N.$$

The best lower bound which holds for all ℓ follows from the construction of regular graphs of large girth by Lubotzky, Phillips, and Sarnak [11], which gives

$$\mathrm{ex}(N, C_{2\ell}) \ge N^{1+(2+o(1))/3\ell}.$$

The correct exponent α_ℓ for which $\mathrm{ex}(N, C_{2\ell}) = N^{\alpha_\ell+o(1)}$ is known only for $\ell = 2, 3, 5$, when it is equal to $1 + 1/\ell$ (see the survey of Füredi and Simonovits [9] and references therein), and finding it for every ℓ is one of the major open problems in extremal graph theory. Can it become easier when we allow the length of an even cycle to grow with N? This paper was inspired by this question. However, instead of the original problem we consider its, nearly equivalent, partition version, and, instead of $\mathrm{ex}(N, C_{2\ell})$, we study

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the Ramsey number $R(C_{2\ell}, K_{1,n})$. Although typically we define Ramsey numbers using colours, here and below we rather view $R(C_{2\ell}, K_{1,n})$ as the minimum N such that each graph on N vertices and minimum degree at least $N-n$ contains a copy of $C_{2\ell}$, i.e. we assume that in the second colour there are no stars $K_{1,n}$ and concentrate on the graph induced by the first colour. Using this point of view let us note that if $\mathrm{ex}(N,H) \le M$, then the complement of any H-free graph on N vertices contains a vertex of degree at least $N - \lfloor 2M/N \rfloor$, which we can express as

$$R(H, K_{1,N-\lfloor 2M/N \rfloor}) \le N \,.$$

On the other hand, if there exists a d-regular H-free graph on N vertices, then, clearly,

$$R(H, K_{1,N-d}) > N \,.$$

Consequently, from the result of Bukh and Jiang and the construction of Lubotzky, Phillips, and Sarnak mentioned above we get

$$n + 2n^{(2+o(1))/3\ell} \le R(C_{2\ell}, K_{1,n}) \le n + 161\sqrt{\ell}\ln \ell n^{1/\ell} + 22\ell^2. \tag{1}$$

Since a graph on N vertices with minimum degree at least $N/2$ is hamiltonian (Dirac [5]), and if its minimum degree is larger than $N/2$, it is pancyclic (Bondy [1]), for $\ell \ge n \ge 2$, we have $R(C_{2\ell}, K_{1,n}) = 2\ell$. Moreover, Zhang, Broersma, and Chen [14] showed that if $n/2 < \ell < n$ then $R(C_{2\ell}, K_{1,n}) = 2n$, while for $3n/8 + 1 \le \ell \le n/2$, we get $R(C_{2\ell}, K_{1,n}) = 4\ell - 1$. Our main result determines the value of $R(C_{2\ell}, K_{1,n})$ for all large ℓ, and n not much larger than ℓ.

Theorem 1. *For every $t \ge 2$, $\ell \ge (19.1t)^9$, and n such that $(t-1)(2\ell-1) \le n-1 < t(2\ell-1)$, we have*

$$R(C_{2\ell}, K_{1,n}) = f_t(\ell, n) + 1,$$

where

$$f_t(\ell, n) = \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\}.$$

The condition $\ell \ge (19.1t)^9$ in Theorem 1 above, which holds when, say, $n \le 0.1\ell^{10/9}$, follows from the bounds given by Lemma 5 below, one of the key ingredients of our argument. It is certainly far from being optimal and we suspect that the result holds for any n growing polynomially with ℓ, but it is conceivable that it remains true even for n which grows exponentially with ℓ. On the other hand, because of (1), the assertion of Theorem 1 fails for, say, $n \ge \ell^{2\ell}$.

We remark that one can use a similar technique to find the value of $\mathrm{ex}(N, C_{2\ell})$ when N is not much larger than ℓ. In this case $\mathrm{ex}(N, C_{2\ell})$ is the same as $\mathrm{ex}(N, C_{\ge 2\ell})$, i.e. the maximum number of edges in a graph on N vertices which contains no cycles of length *at least* 2ℓ. Let us recall that $\mathrm{ex}(N, C_{\geq 2\ell})$, was determined already by Erdős and Gallai [6] who also described the structure of all extremal graphs for this problem – it turns out that all their blocks, except perhaps one, are cliques on $2\ell - 1$ vertices. However, the behaviour of $R(C_{2\ell}, K_{1,n})$ seemed to us more intriguing. Indeed, for a given ℓ and $(t-1)(2\ell-1) \leq n-1 < \frac{t^2}{t+1}(2\ell-1)$ we have

$$f_t(\ell, n) = (2\ell-1)t,$$

i.e. for this range of n the value of $R(C_{2\ell}, K_{1,n})$ does not depend on the size of the star. On the other hand, as is shown in the next section, for $\frac{t^2}{t+1}(2\ell-1) \leq n-1 < t(2\ell-1)$, when

$$f_t(\ell, n) = n + \lfloor (n-1)/t \rfloor$$

the 'extremal graphs' which determine the value of $R(C_{2\ell}, K_{1,n})$ typically have all blocks smaller than $2\ell - 1$.

2. The lower bound for $R(C_{2\ell}, K_{1,n})$

In this section we show that for given integers t, ℓ, and n such that $(t-1)(2\ell-1) \leq n-1 < t(2\ell-1)$, we have

$$R(C_{2\ell}, K_{1,n}) > f_t(\ell, n) = \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\}. \tag{2}$$

Let us consider first the graph H_1 which consists of t vertex-disjoint copies of the complete graph $K_{2\ell-1}$. Clearly, $|V(H_1)| = t(2\ell-1)$ and $H_1 \not\supseteq C_{2\ell}$. Moreover, $\Delta(\overline{H}_1) = (t-1)(2\ell-1) \leq n-1$ yielding $\overline{H}_1 \not\supseteq K_{1,n}$. Hence

$$R(C_{2\ell}, K_{1,n}) > t(2\ell-1)\,.$$

Now let $k = n-1-t\lfloor (n-1)/t \rfloor$ and $m = \lfloor (n-1)/t \rfloor + 1$. We define a graph H_2 as a union of k vertex-disjoint complete graphs K_m and $t+1-k$ other copies of K_m which are 'almost' vertex-disjoint except that they share exactly one vertex (see Figure 1).

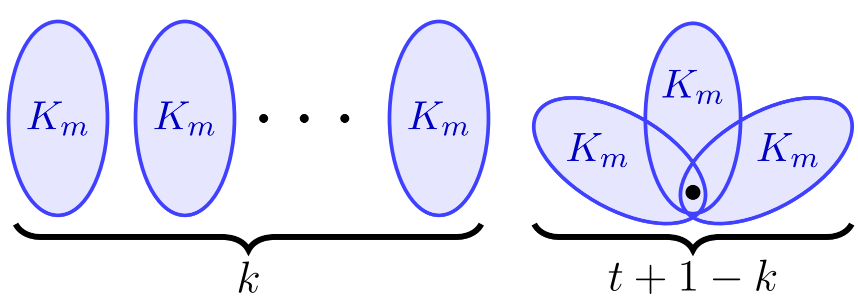

Figure 1. Graph H_2. Here $k = n-1-t\lfloor \frac{n-1}{t} \rfloor$ and $m = \lfloor \frac{n-1}{t} \rfloor + 1$

Then

$$\begin{aligned}|V(H_2)| &= km + (t+1-k)(m-1) + 1 = (t+1)m - (t-k)\\ &= (t+1)(\lfloor (n-1)/t\rfloor + 1) - t + n - 1 - t\lfloor (n-1)/t\rfloor\\ &= n + \lfloor (n-1)/t\rfloor.\end{aligned}$$

Note also that $n-1 < t(2\ell-1)$, and so $m = \lfloor (n-1)/t\rfloor + 1 \le 2\ell - 1$. Hence $H_2 \not\supseteq C_{2\ell}$. Finally,

$$\Delta(\overline{H}_2) = |V| - m = n + \lfloor (n-1)/t\rfloor - \lfloor (n-1)/t\rfloor - 1 = n-1.$$

Therefore

$$R(C_{2\ell}, K_{1,n}) > |V(H_2)| = n + \lfloor (n-1)/t\rfloor,$$

and (2) follows.

Let us remark that the two graphs H_1 and H_2 we used above are by no means the only 'extremal graphs' with $R(C_{2\ell}, K_{1,n}) - 1$ vertices. Let us take, for example, $n = 4.1\ell$. Then $R(C_{2\ell}, K_{1,n}) = 3(2\ell-1)+1$ and the lower bound for $R(C_{2\ell}, K_{1,n})$ is 'certified' by the graph H_1' which consists of three vertex disjoint cliques $K_{2\ell-1}$. However, if we replace each of these cliques by a graph on $2\ell-1$ vertices and minimum degree 1.91ℓ, the complement of the resulting graph will again contain no $K_{1,n}$, so each such graph shows that $R(C_{2\ell}, K_{1,n}) > 3(2\ell-1)$ as well. On the other hand, adding to H_1' a triangle with vertices in different cliques does not result in a copy of $C_{2\ell}$, so H_1' is not even a maximal extremal graph certifying that $R(C_{2\ell}, K_{1,n}) > 3(2\ell-1)$.

3. Cycles in 2-connected graphs

In order to show the upper bound for $R(C_{2\ell}, K_{1,n})$ we have to argue that large graphs with a sufficiently large minimum degree contain $C_{2\ell}$. In this section we collect a number of results on cycles in 2-connected graphs we shall use later on.

Let us recall first that the celebrated theorem of Dirac [5] states that each 2-connected graph G on n vertices contains a cycle of length at least $\min\{2\delta(G), n\}$, and, in particular, each graph with minimum degree at least $n/2$ is hamiltonian. Below we mention some generalizations of this result. Since we are interested mainly in even cycles, we start with the following observation due to Voss and Zuluaga [13].

Lemma 2. *Every 2-connected graph G on n vertices contains an even cycle C of length at least* $\min\{2\delta(G), n-1\}$. □

The following result by Bondy and Chvátal [2] shows that the condition $\delta(G) \ge n/2$, sufficient for hamiltonicity, can be replaced by a somewhat weaker one. Recall that the closure of a graph $G = (V, E)$ is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least $|V|$ until no such pair remains.

Lemma 3. *A graph G is hamiltonian if and only if its closure is hamiltonian.* □

If we allow $\delta(G) > n/2$, then, as observed by Bondy [1], G becomes pancyclic. We use the following strengthening of this result, proved under slightly stronger assumptions, due to Williamson [12].

Lemma 4. *Every graph $G = (V, E)$ on n vertices with $\delta(G) \geq n/2 + 1$ has the following property. For every $v, w \in V$ and every k such that $2 \leq k \leq n - 1$, G contains a path of length k which starts at v and ends at w. In particular, G is pancyclic.* □

Finally, we state a theorem of Gould, Haxell, and Scott [10], which is crucial for our argument. Here and below $\mathrm{ec}(G)$ denotes the length of the longest even cycle in G.

Lemma 5. *Let $a > 0$, $\hat{K} = 75 \cdot 10^4 a^{-5}$, and G be a graph with $n \geq 45\hat{K}/a^4$ vertices and minimum degree at least an. Then for every even $r \in [4, ec(G) - \hat{K}]$, G contains a cycle of length r.*

Let us also note the following consequence of the above results.

Lemma 6. *For $c \geq 1$ we set*

$$K(c) = 24 \cdot 10^6 c^5 = 75 \cdot 10^4 (1/2c)^{-5}, \tag{3}$$

and let $\ell \geq 360c^4 K(c)$. Then for every 2-connected $C_{2\ell}$-free graph $H = (V, E)$ such that $|V| \leq 2\ell c$ and $\delta(H) \geq \ell + K(c)$, we have

$$|V| \leq 2\ell - 1.$$

Proof. Let us consider first the case $|V| < 2\ell + 2K(c) - 2$. Then, since

$$\delta(H) \geq \ell + K(c) > |V|/2 + 1,$$

from Lemma 4 we infer that H is pancyclic. But $C_{2\ell} \nsubseteq H$ meaning that $|V| \leq 2\ell - 1$, as required.

On the other hand, for $|V| \geq 2\ell + 2K(c) - 2$ Lemma 2 implies that

$$\mathrm{ec}(H) \geq 2\ell + 2K(c) - 2 > 2\ell + K(c)$$

Moreover, as $|V| \leq 2\ell c$ and $\ell \geq 360c^4 K(c)$, one gets

$$\delta(H) > \ell \geq \frac{1}{2c}|V| \quad \text{and} \quad |V| > 2\ell \geq 45 \left(\frac{1}{2c}\right)^{-4} K(c).$$

Therefore, from Lemma 5 applied to H with $a = 1/(2c)$, we infer that H contains a cycle of length 2ℓ, contradicting $C_{2\ell}$-freeness of H. □

4. Proof of the main result

The two examples of graphs we used to verify the lower bound for $R(C_{2\ell}, K_{1,n})$ (see Section 2) suggest that a natural way to deal with the upper bound for $R(C_{2\ell}, K_{1,n})$ is to show first that each $C_{2\ell}$-free graph G with a large minimum degree has all blocks smaller than 2ℓ. However, most results on the existence of cycles in 2-connected graphs use the minimum degree condition, and even if the minimum degree of G is large, some of its blocks may contain vertices of small degree. Nonetheless we shall prove that the set of vertices in each such G contains a 'block-like' family of 2-connected subgraphs without vertices of very small degree. Then, based on the results of the last section, we argue that each subgraph in such family is small. In the third and final part of our proof we show that if this is the case, then G has at most $f_t(\ell, n)$ vertices.

Before the proof of Theorem 1 we state two technical lemmata. The first one will become instrumental in the first part of our argument, when we decompose the graph G into 2-connected subgraphs without vertices of small degree.

Lemma 7. *Let $n \geq k \geq 2$. For each graph G with n vertices and minimum degree $\delta(G) \geq n/k + k$, there exists an $s < k$ and a set of vertices $U \subset V(G)$, $|U| \leq s - 1$, such that $G - U$ is a union of s vertex-disjoint 2-connected graphs.*

Proof. Consider a sequence $U_0, U_1, \ldots, U_t = U$ of subsets of V which starts with $U_0 = \emptyset$ and, if $G - U_i$ contains a cut vertex v_i, we put $U_{i+1} = U_i \cup \{v_i\}$. The process terminates when each component of $G - U_i$ is 2-connected. Note that in each step the number of components of a graph increases by at least one, so $G - U_i$ has at least $i + 1 = |U_i| + 1$ components. Moreover, the process must terminate for $t < k - 1$ since otherwise the graph $G - U_{k-1}$ would have $n - k + 1$ vertices, at least k components, and the minimum degree at least $n/k + 1$ which, clearly, is impossible. Hence the graph $G - U = G - U_t$ has $n - t$ vertices, $s \geq |U| + 1 = t + 1$ components, and minimum degree larger than $n/k + 1$. Finally, let us notice that, again, since each component has more than n/k vertices, we must have $s < k$. □

The following result is crucial for the final stage of our argument, when we show that each graph G with a large minimum degree, which admits a certain block-like decomposition into small 2-connected subgraphs, cannot be too large.

Lemma 8. *For a given set V and positive integers $\ell, s, t, n \geq 2$, satisfying $(t-1)(2\ell-1) \leq n - 1 < t(2\ell - 1)$, let $V_1, V_2, \ldots, V_s$ be subsets of V such that*

(*i*) $V = V_1 \cup V_2 \cup \cdots \cup V_s$,

(*ii*) $|V_i| \leq 2\ell - 1$ *for* $i = 1, 2, \ldots, s$, *(iii)* $|V \setminus V_i| \le n-1$ *for* $i = 1, 2, \dots, s$,

(iv) $|V_1| + |V_2| + \cdots + |V_s| \le |V| + s - 1$.

Then

$$|V| \le f_t(\ell, n) = \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\}.$$

Proof. Note first that if $s \le t$, then *(i)* and *(ii)* imply that $|V| \le t(2\ell - 1)$. Thus, let us assume that $s \ge t+1$. Then,

$$\begin{aligned} s(n-1) &\overset{(iii)}{\ge} \sum_{i=1}^{s} |V \setminus V_i| = s|V| - (|V_1| + |V_2| + \cdots + |V_s|) \\ &\overset{(iv)}{\ge} s|V| - (|V| + s - 1) = (s-1)|V| - (s-1), \end{aligned}$$

and thereby

$$|V| \le \frac{s}{s-1}(n-1) + 1 = n + \frac{n-1}{s-1} \le n + \frac{n-1}{t}.$$

Since $|V|$ is an integer, the assertion follows. □

Proof of Theorem 1. Since we have already bounded $R(C_{2\ell}, K_{1,n})$ from below in Section 2, we are left with the task of showing that

$$R(C_{2\ell}, K_{1,n}) \le f_t(\ell, n) + 1.$$

For this purpose, let $t \ge 2$,

$$\ell \ge (19.1t)^9 > 360(t+1)^4 \cdot K(t+1),$$

where $K(t+1) = 24 \cdot 10^6 (t+1)^5$ is the function defined in (3), and

$$(t-1)(2\ell - 1) \le n - 1 < t(2\ell - 1).$$

Moreover, let $G = (V, E)$ be a $C_{2\ell}$-free graph on

$$|V| = f_t(\ell, n) + 1$$

vertices such that $\overline{G} \not\supseteq K_{1,n}$ (or equivalently, $\Delta(\overline{G}) \le n-1$).

Recall that $f_t(\ell, n) = \max\{t(2\ell-1), n + \lfloor (n-1)/t \rfloor\}$ and observe that

$$(n-1) + \frac{t(2\ell-1)}{t+1} < f_t(\ell, n) < (t+1)(2\ell-1). \tag{4}$$

Indeed, the upper bound follows immediately from the fact that $n - 1 < t(2\ell - 1)$, so it is enough to verify the lower bound for $f_t(\ell, n)$. If

$$(n-1) + \frac{t(2\ell-1)}{t+1} < t(2\ell-1)$$

then we are done, otherwise we have

$$\frac{t(2\ell-1)}{t+1} \le \frac{n-1}{t}$$

and, since $f_t(\ell,n) \ge n + \lfloor \frac{n-1}{t} \rfloor$, (4) holds as well.

Our aim is to show that G contains a family of 2-connected subgraphs $G_i = (V_i, E_i)$, $i = 1, 2, \ldots, s$, such that their vertex sets fulfil the conditions (*i*)-(*iv*) listed in Lemma 8.

We first apply Lemma 7 to G with $k = \frac{(t+1)^2+1}{t}$. We are allowed to do this, because (4) tells us that

$$\delta(G) = |V| - 1 - \Delta(\bar{G}) \ge f_t(\ell,n) - (n-1) > \frac{t(2\ell-1)}{t+1} \ge \frac{t|V|}{(t+1)^2} \tag{5}$$

However, both $|V|$ and ℓ are much larger than t, in particular, $|V| \ge 2\ell > (19.1t)^9$. Hence,

$$\delta(G) \ge \frac{t|V|}{(t+1)^2} > \frac{t}{(t+1)^2+1}|V| + \frac{(t+1)^2+1}{t}$$

and the assumptions of Lemma 7 hold with $k = \frac{(t+1)^2+1}{t} \le t+3$. Thus, there exists $s \le t+2$ and a set of vertices $U \subset V$, $|U| \le s-1$, such that $G - U$ is a union of s vertex-disjoint, 2-connected graphs, $G'_i = (V'_i, E'_i)$. Note that since $|U| \le t+1$ and $\ell > 4K(t+1)$ are large,

$$\delta(G'_i) \ge \delta(G) - |U| > \frac{2(2\ell-1)}{3} - (t+1) > \ell + K(t+1). \tag{6}$$

Moreover, clearly, $|V'_i| \le |V| < (t+1)2\ell$, so Lemma 6 applied to G'_i, with $c = t+1$, gives

$$|V'_i| \le 2\ell - 1 \quad \text{for} \quad i = 1, 2, \ldots, s.$$

Now, for every $i = 1, 2, \ldots, s$, we define

$$U_i = \{u \in U : \deg_G(u, V'_i) \ge 4t\}, \quad V_i = V'_i \cup U_i, \quad \text{and} \quad G_i = G[V_i].$$

We will show that the sets $V_1, V_2, \ldots, V_s$ satisfy the conditions (*i*)-(*iv*) of the hypothesis of Lemma 8.

In order to verify (*i*) observe that since the minimum degree of G is large, i.e. $\delta(G) \ge 8t^2$, every vertex $u \in U$ belongs to at least one of the sets U_i, and therefore $V = V_1 \cup V_2 \cup \cdots \cup V_s$.

To prove that $|V_i| \le 2\ell - 1$, let us assume that $|V_i| \ge 2\ell$. Now take any subset $\hat{U}_i$ of U_i, with $|\hat{U}_i| = 2\ell - |V'_i|$ elements and set $H_i = G[V'_i \cup \hat{U}_i]$. Note that H_i has 2ℓ vertices. We will argue that H_i is hamiltonian. To this end, consider the closure of H_i. From (6) we know that all vertices from V'_i have degree at least $\delta(G'_i) > \ell + K(t+1)$, so in the closure of H_i the set V'_i spans a clique of size at least $2\ell - |U| \ge 2\ell - t - 1$. On the other hand, each vertex from $\hat{U}_i$ has in V'_i at least $4t$ neighbours, so the closure of H_i is the complete graph and therefore, by Lemma 3, H_i is hamiltonian. However it means that

$C_{2\ell} \subseteq H_i \subseteq G$ which contradicts our assumption that G is $C_{2\ell}$-free. Consequently, for every $i = 1, 2, \ldots, s$, we have $|V_i| \le 2\ell - 1$, as required by *(ii)*.

Note that from (6) it follows that $|V_i'| > \delta(G_i') > \ell$. Since $U \setminus U_i$ sends at most $4t|U| \le 4t(t+1) < \ell$ edges to the set V_i', there exists a vertex $v_i \in V_i' \subseteq V_i$ which has all its neighbours in G_i. It means however that, since $\overline{G} \not\supseteq K_{1,n}$, the set $V \setminus V_i$, which contains only vertices which are not adjacent to v_i, has at most $n-1$ elements, and so *(iii)* holds.

Finally, to verify *(iv)* consider an auxiliary bipartite graph $F = (V_F, E_F)$, where $V_F = \{V_1', V_2', \ldots, V_s'\} \cup U$ and

$$E_F = \{uV_i' : u \in U_i\}.$$

We claim that F is a forest. Indeed, assume for the sake of contradiction that F contains a cycle $C = V_{i_1}' u_{j_1} \ldots V_{i_w}' u_{j_w} V_{i_{w+1}}'$, $i_1 = i_{w+1}$. Observe that every vertex u_{j_x}, $x = 1, 2, \ldots, w$, has at least two neighbours in both sets V_{i_x}' and $V_{i_{x+1}}'$. Moreover, $\delta(G_i') > \ell + 1$ and $|V_i'| \le 2\ell - 1$, so from Lemma 4 it follows that any two vertices of V_i' can be connected by a path of length y for every $y = 2, 3, \ldots, |V_i| - 1$. Therefore, since $w \le |U| \le t+1 \le \ell/4$, the existence of C in F implies the existence of a cycle $C_{2\ell}$ in G, contradicting the fact that G is $C_{2\ell}$-free.

Since F is a forest it contains at most $|U| + s - 1$ edges, i.e.

$$\sum_{u \in U} \deg_F(u) \le |U| + s - 1.$$

Note that in the sum $|V_1| + |V_2| + \cdots + |V_s|$ each vertex from $\bigcup_i V_i' = V \setminus U$ is counted once, and each vertex $u \in U$ is counted precisely $\deg_F(u)$ times, so

$$|V_1| + \cdots + |V_s| = |V| - |U| + \sum_{u \in U} \deg_F(u) \le |V| + s - 1,$$

as required by *(iv)*.

Now we can apply Lemma 8 and infer that $|V| \le f_t(\ell, n)$ while we have assumed that $|V| = f_t(\ell, n) + 1$. This final contradiction completes the proof of the upper bound for $R(C_{2\ell}, K_{1,n})$ and, together with (2), concludes the proof of Theorem 1. □

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