**Parthasarathy, shift-compactness and infinite combinatorics**

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**Abstract.** Parthasarathy's heritage has a hidden component arising from a variant of his concept of shift-compactness which yields quick proofs of fundamental theorems reviewed here. We demonstrate the closeness of the variant notion to his original one as arising in the *tacit* treatment of all possible convergent centrings.

We also include a very short proof of the Effros Mapping Theorem – a non-linear version of the Open Mapping Theorem. This is deduced from a shift-compactness theorem. Both these can be given a constructive form by implementing a constructive improvement to a theorem on the separation of points and closed nowhere dense sets.

**Keywords.** Steinhaus-Weil property, shift-compactness, infinite combinatorics, amenability at 1, measure subcontinuity.

**Classification**: Primary 22A10, 43A05; Secondary 28C10.

## 1. Probabilistic results

*Central limit theory*

Parthasarathy introduced the concept of shift-compactness for probabilistic reasons. We recall the classical central limit problem in probability theory. The prototype here is the original central limit theorem (CLT): that if one has a sequence of independent copies $X_n$ of a (real-valued) random variable $X$ with finite mean $\mu$ and variance $\sigma^2$, then centring at means and scaling gives convergence in law of their sum to the standard Gaussian (or normal) law, $\Phi = N(0,1)$:

$$\mathbb{P}((\sum_1^n X_i - n\mu)/(\sqrt{n}\sigma) \le x) \to \Phi(x) \qquad (n \to \infty) \qquad \forall x.$$

This can be vastly generalised, to suitable ('uniformly asymptotically negligible', uan) doubly-indexed arrays $(X_{nk})$. The sequence of row-sums of the array are approximated by 'accompanying infinitely divisible laws', which have the same limits. These limits are the *infinitely divisible laws*, described by the *Lévy-Khintchine* formula. This is the theme of the classic book [GneK1954]. The centring (at means when they exist, though they may not) is the "shift in shift-compactness".

The area is best seen from a stochastic-process viewpoint, where the infinitely divisible limit laws are generalised to *Lévy processes*, for which see e.g. Bertoin [Ber1996]. Here one has a continuous-time parameter in the limit process, while the arrays one begins with are discrete-parameter. (For more on the interplay between discrete-space and continuous-parameter results, see the companion paper [BinS].) To embed the initial discrete setting in the continuous final one, one needs *embedding* theorems; see [Hey1977, Ch. III]. Embedding fails in the general locally compact case (see below).

As usual with limit theorems, exact independence is not needed, but can be weakened to suitable forms of *weak dependence*, in particular to a hierarchy of *mixing conditions*. For details, see e.g. [EbeT1986], [Bra2007].

The theory was extended from the line $\mathbb{R}$ to locally compact abelian groups with a countable basis for their topology by Parthasarathy [Par1967, Ch. IV] (cf. Parthasarathy and Schmidt [ParS1972], Guichardet [Gui1972]), and Heyer [Hey1977, Ch. V], using Parthasarathy's concept of *shift-compactness* (below). For the failure to extend to general locally compact groups, see [Hey1977, Ch. VI]. To give a sample result: the condition for embeddability to hold for a locally compact group $G$ is that *the connected component $G_0$ of the identity $1_G$ be locally arcwise connected* ([Hey1977, Th. 3.5.12]).

*Shifts, stationarity and Szegő*

Shifts are ubiquitous and of fundamental importance. In particular, they are crucial in spectral function theory, for which see Nikolskii [Nik1986]. Szegő theory is of fundamental importance in prediction theory, for which see e.g. [Bin2012]. Stationarity is important in both areas.

*Other areas of Parthasarathy's work*

*Quantum probability*

The continuous tensor products of [ParS1972] (see e.g. its review MR3907334) are important in the field of *quantum probability*, one of Parthasarathy's major lifelong interests. For textbook expositions see [Par1992], Meyer [Mey1993]. See also [Par1988a], [Par1992], [Par1994],[Par2018], [BarGJ2001], [BarGJ2003], [Bin2003].

*Combinatorics and graph theory*

Parthasarathy worked extensively on combinatorics and graph theory, particularly in his early work (perhaps stemming from his 1962 PhD under C. R. Rao at the Indian Statistical Institute, Kolkata). These areas are important in statistics, in design of experiments.

*Parthasarathy on Kolmogorov*

We conclude here by referring to Parthasarathy's excellent obituary article on Kolmogorov [Par1988b], which is full of interesting reminiscences of this greatest of all probabilists.

**2. Groups, Baire spaces and analyticity**

Below we develop shift-compactness notions related to that established by Parthasarathy. The context here will always be that of a *separable metric group* $G$. The group $G$ may act on a topological space $X$. Hereafter such a space $X$ usually carries a metric $d^X$ (so is metric, is metrisable), but on occasion we may permit a more general topological structure, and this will be stated explicitly. In this connection we recall the Birkhoff-Kakutani invariant metrization result, which was also key in his work ([Par1967, Ch. III Lemma 4.1, referring to [Kel1955, Ch. 6 p. 210]). We formulate this in the style of Kolmogorov normability, by reference to a (group) norm $||x||$, one which satisfies the three axioms:
i) Positivity: $||x|| > 0$ unless $x = 1_G$, the identity element of $G$;
ii) Symmetry: $||x^{-1}|| = ||x||$, i.e. inversion invariance;
iii) Subadditivity: $||xy|| \leq ||x|| + ||y||$, i.e. the triangle inequality.

**Theorem 2.1** [Birkhoff-Kakutani Normability Theorem, [Bir1936], [Kak1936]]. *A first-countable right-topological group $X$ is a normed group iff inversion and multiplication are continuous at the identity.*

Here first-countable means that the identity of the group has a countable basis for its neighbourhoods; right-topological means that multiplication on the right is continuous. Thus a normed group need not be a topological group (see below). Such a norm generates a left- and a right-invariant metric, e.g. $d_R(x, y) = ||xy^{-1}||$. Thus we denote open balls[1] of radius $\varepsilon > 0$ centred at $1_G$ by $B_\varepsilon(1_G)$. A further tool is provided by two inter-related concepts due to Baire. We recall that a subset is *meagre* if it is a countable union of nowhere dense sets. *Baire's Category Theorem* asserts that a complete metric space is non-meagre; equivalently, a countable intersection of dense open sets is a dense $\mathcal{G}_\delta$. A (separable!) group $X$ is *Baire*, meaning that the conclusion in Baire's theorem holds in it, iff some open subset is non-meagre (equivalently, all non-empty open subsets are such). A subset $T \subseteq X$ *has the Baire property* (is *Baire*, for short) if $T$ is open modulo a meagre set; the relevant open set is referred to as its *quasi-interior*. (It is non-empty iff $T$ is non-meagre.)

[1] [Par1967] uses sphere in the sense of ball, sometimes open, sometimes closed; its boundary there is denoted by B (see page 50).

The concept of being meagre relies on countability. However, Banach's ingenious use of an open disjoint 'almost covering' (i.e. omitting only a nowhere dense set) led to *Banach's Category Theorem,* according to which a union of *any* family of meagre open sets is meagre. (The result is thus sometimes called the Banach Union Theorem.) This is very much in the same spirit as $\tau$-additivity in measure theory (see e.g. [Fre2003]). A measure $\mu$ is *$\tau$-additive* if, for any increasing (perhaps uncountable) family of sets, $\{A_\lambda : \lambda < \kappa\}$ say (well-ordered by ordinals),

$$\sup_{\lambda<\kappa} \mu(A_\lambda) = \mu(\sup_{\lambda<\kappa} A_\lambda), \text{ i.e. } = \mu\left(\bigcup_{\lambda<\kappa} A_\lambda\right).$$

Baire's Theorem has a countability-free form in the statement below, which, however, refers to the *countable chain condition*, i.e. that any family of disjoint open sets is at most countable (cf. [Fre1984, 11A]).
*Martin's Axiom, MA.*

*Any compact Hausdorff space satisfying the countable chain condition is not the union of less than continuum many nowhere dense subsets.*

This is a topological variant of Martin's Axiom, which is usually stated in its original context of partial orders. For a proof of the equivalence of the two variants, and a discussion of its relative consistency, see e.g. Weiss [Weis1984]. (The axiom reduces to Baire's Theorem if the continuum is $\aleph_1$.) For a discussion of the axiom in its original context and its consequences see [Fre1984]; the topological aspects are commented on ibid. §43, p. 192.

We further recall that a subset of $X$ is *analytic* if it is the continuous image of a Polish space (complete separable metric space). A continuous image of an analytic set is manifestly analytic. Relevance here rests on the fact that analytic subsets are Baire (by a result due to Nikodym, see e.g. [RogJ1980, §2.9]), and so analyticity is the means by which the Baire property is preserved under continuous images.

Finally, we note that situations below will arise which divide into two sharply contrasting cases – usually where behaviour is, in some appropriate sense, either very good or very bad. Thus in the dichotomies of §5 a subgroup is either meagre or the whole group. In the Darboux dichotomy of §3, a real-valued additive function on $\mathbb{R}$ is either continuous or locally unbounded on every non-meagre Baire subset of $\mathbb{R}$ (cf. [BinJJ2020]). Such dichotomy often has a generic aspect: if something works at all, it works nearly everywhere.

The source of this genericity is addressed below: a property inheritable by supersets either holds generically or fails outright. The nub is the following straightforward result. Below $\mathcal{B}a = \mathcal{B}a(X)$ denotes the Baire subsets of $X$.

**Theorem 2.2** [Generic Dichotomy Principle, [BinO2010b]]. *For $F : \mathcal{B}a \to \mathcal{B}a$ monotonic, i.e. satisfying $F(S) \subseteq F(T)$ for $S \subseteq T$, either:*
(i) *there is a non-meagre $S \in \mathcal{B}a$ with $S \cap F(S) = \varnothing$, or,*
(ii) *for every non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of $T$.*

*Equivalently: the existence condition that $S \cap F(S) \neq \varnothing$ should hold for all non-meagre $S \in \mathcal{B}a$, implies the genericity condition that, for each non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi almost all of $T$.*

### 3. Shift-compactness and infinite combinatorics

We first recall an important tool of [Par1967], the notion of shift-compactness for probability measures and two related key properties. The context is the convolution semi-group $\mathcal{M}(X)$ of (Borel, probability) measures on a separable metric *group* $X$. This means that $\mathcal{M}(X)$, understood to be equipped with the weak topology, may itself be equipped with a separable metric [Par1967, II Th. 6.2], to which the Birkhoff-Kakutani theorem of the preceding section applies, in view of the continuity properties of convolution.

The subset $K \subseteq \mathcal{M}(X)$ is *right shift-compact* if for any sequence of measures $\mu_n \in K$ there are right translations $t_n \in X$ such that the convolution sequence $\mu_n * t_n$ has a convergent subsequence $\mu_{n_m} * t_{n_m} \Rightarrow \lambda \in \mathcal{M}(X)$. Similarly for left shift-compactness.

Using continuity of $*$ in the special case $\mu_{n_m} * t \Rightarrow \lambda$ gives $\mu_{n_m} \Rightarrow \lambda * t^{-1}$, showing $\mu_n$ to be conditionally compact. This holds more generally:

**Theorem 3.1** [Par1967, III.Th. 2.1]. *If both sequences of measures $\mu_n$ and $\lambda_n = \mu_n * \nu_n$ are conditionally compact, then so also is $\nu_n$.*

The result applies to the special case $\nu_n = \delta_{t_n}$ of Dirac measures with $t_n$ as above, showing that $t_{n_m}$ is convergent (to $t$, say). More general adjustments of a sequence $\mu_n$ towards conditional compactness emerge as a disguised translational adjustment in the following result.

**Theorem 3.2** [Par1967, III.Th. 2.2]. *If the sequence of measures $\lambda_n = \mu_n * \nu_n$ is conditionally compact, then $\mu_n$ and $\nu_n$ are respectively right and left shift-compact.*

The concept of shift-compactness (often narrowed down to $K$ a sequence) is motivated by limit theorems and the detailed study of Gaussianity in $\mathcal{M}(X)$. See §1 and, for instance, [Par1967, p.80, pp. 113-4] (also pp. 171, 182,187,195).

**Theorem 3.3.** *If $\mu_m * t_m \Longrightarrow \lambda \in M(X)$ with $t_m \to t$, then for each $\lambda$-non-null Borel set $A$ with $\lambda$-null boundary there is a sequence $u_m$ so that*

$$\{m \in \mathbb{N} : u_m t_m \in A\} \text{ is infinite.}$$

*In particular, in the abelian case, for $t_m = tz_m$ with $z_m \to 1_X$ and $s_m = tu_m$*

$$\{m \in \mathbb{N} : s_m z_m \in A\} \text{ is infinite.}$$

**Proof.** By the Portmanteau Theorem [Bil1999, Th. 2.1] (cf. [Par1967, Ch. II Th. 6.1]) $\mu_m * t_m(A) > 0$ infinitely often (i.o), i.e.

$$0 < \mu_m * t_m(A) = \int_{u \in At_m^{-1}} d\mu_m(u) \text{ i.o.}$$

So, for infinitely many $m$, there is some $u_m$ with $u_m t_m \in A$. In the abelian case, writing $t_m = tz_m$ with $z_m \to 1_X$ and $s_m = tu_m$ yields the desired result. □

Observe that the final assertion that $s_m z_m \in A$ i.o. yields an embedding of an infinity of terms of apparently just the *one* null sequence $z_m$ into many sets $A$ associated with the Borel measure $\lambda$. However, Parthasarathy's definition *tacitly* admits a *plurality* of convergent subsequences (to appropriate limit measures $\lambda$, and/or their translates). This motivates a related shift-compactness concept, now in a metric group, albeit focused not on one useful null sequence $z_m \to 1_X$, but on *all* null sequences in $X$.

**Definition.** Say that an arbitrary ('target') subset $T \subseteq X$ is *shift-compact* if for any *null sequence* $z_m \to 1_X$ there is $t \in T$ such that

$$\mathbb{M}_t = \{m \in \mathbb{N} : tz_m \in T\} \text{ is infinite.}$$

In the case of the additive group of reals, such a property was initially studied (albeit under a different name, and specialized to co-finite sets $\mathbb{M}_t$) by Kestelman [Kes1947] and later by Borwein and Ditor [BorwD1978] in the measure case. It is thus appropriate to name the following result on infinite combinatorics after their pioneering work. For its extensive usage see [BinO2024]. Theorems below that assert the shift-compactness of specified sets will be termed shift-compactness theorems.

**Theorem 3.4** [Kestelman-Borwein-Ditor Theorem, Theorem KBD]. *Let $z_n \to 0$ be a null sequence in $\mathbb{R}$. If $T$ is a measurable/Baire subset of $\mathbb{R}$, then for generically all (= almost all/quasi all) $t \in T$ there is an infinite set $\mathbb{M}_t$ (or subsequence) such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

*Thus Baire non-meagre/measurable non-null sets are shift-compact.*
Here a property holds for almost all, resp. quasi all, $t$ if it holds for all $t$ off a null, resp. meagre, set. The genericity here results from the Generic Dichotomy Principle, Theorem 2.2.

**Relation to Parthasarathy shift-compactness.** One may relax the definition above to parallel more closely that of [Par1967] via Theorem 3.3 and (temporarily) call $T \subseteq X$ *dynamically* shift-compact, if for any $z_n \to 1_X$ there are $t_n \in T$ with

$$t_n z_n \in T \text{ i.o., equivalently } z_n \in TT^{-1} \text{ i.o.}$$

This, however, in turn is equivalent to $1_X$ being an interior point of $TT^{-1}$. Indeed, if $B_\varepsilon(1_X) \subseteq TT^{-1}$ for some $\varepsilon > 0$, then in fact $z_m \in TT^{-1}$ for co-finitely many $m$. Conversely, if $1_X$ is not an interior point of $TT^{-1}$, then for $m = 1, 2, ...$ we may select $z_m \in B_{1/m}(1_X) \backslash TT^{-1}$, giving $z_m \to 1_X$ but with $z_m$ not i.o. in $TT^{-1}$. Thus the celebrated Steinhaus-Weil interior point property, SW, of Theorem 9.1 emerges here. In brief, dynamical shift-compactness is just *disguised SW.*

The emphasis here on *all* null sequences in turn prompts an immediate question concerning for which Borel measures $\lambda$ do all $\lambda$-non-null sets $A$ have the embedding property of Theorem 3.3, that $s_m z_m \in A$ i.o. For locally compact groups $X$, this is answered by the Simmons-Mospan Converse to SW, Theorem 9.2, that $\lambda$ is absolutely continuous w.r.t. Haar measure.

Thus the difference between KBD-like shift-compactness and Parthsarathy's lies in admitting all possible null sequences, so may be said to admit all possible convergent centring.

**Remark.** In the proof of Theorem 3.3, passing to a compact $K \subseteq A$ with $\lambda(K) > 0$, given continuity of convolution and so of $t \mapsto \lambda(Kt^{-1})$, a situation could emerge rather like that in [Par1967, Ch. 3 Lem. 3.1] (cited as Lemma 7.2 below) that for some infinite sequence of integers $m(1), ..., m(n), ...$ all the finite intersections $Kt_{m(1)}^{-1} \cap ... \cap Kt_{m(n)}^{-1}$ are non-empty (through being non-null). There would then be $u$ in their intersection such that $u \in Kt_{m(n)}^{-1}$, yielding $utz_{m(n)} \in K \subseteq A$, exactly as with the KBD-like shift-compactness. Evidently, having $t_n \equiv t$ (down a subsequence) retrieves the narrower (static) definition.

The dynamic version neverthlesss has its merits as an operational convenience. Thus, recall Darboux's Theorem [BinO2011] (cf. [BinJJ2020]) that an additive function bounded on an interval is continuous. A strengthening of this theorem may be obtained by an argument similar to the 'reduction to SW' above, to show that if $f : \mathbb{R} \to \mathbb{R}$ is additive and bounded (in modulus) by $M$ on the dynamically shift-compact set $T$, then $f$ is continuous, and so linear. Indeed, $f$ will be bounded on any interval, as otherwise we may choose a convergent sequence $x_n \to x$ with, say, $f(x_n) \geq n$. Writing $x_n = x + z_n$ with $z_n \to 0$, choose $t_n \in T$ with $t_n + z_n \in T$ i.o. Then i.o.

$$n \leq f(x_n) = f(x) + f(z_n + t_n) - f(t_n) \leq f(x) + 2M,$$

which gives a contradiction for large enough $n$.

Almost the same argument yields a proof of a standard result of functional analysis, the Banach-Steinhaus Uniform Boundedness Theorem ([Rud1991, §2.3], [Con1990, III §14, 14.1]).

**Theorem 3.5**. *For $X$ a non-meagre topological vector space and $F$ a family of continuous linear functionals, if for each $x$ the set $\{||f(x)|| : f \in F\}$ is bounded, then $\{||f(x)|| : f \in F\}$ is bounded on a nhd of 0.*

**Proof.** Suppose otherwise. Then, for each $n$, there exist $x_n \in X$ and $f_n \in F$ such that $||f_n(x_n)|| \geq n$. As $f$ is continuous, $\{x : ||f(x)|| \leq n\}$ is closed, and so is

$$A_n := \bigcap\nolimits_{f \in F} \{x : ||f(x)|| \leq n\},$$

so has the Baire property. By assumption

$$X = \bigcup\nolimits_n A_n.$$

So, since $X$ is non-meagre, there is $N$ such that $A_N$ is non-meagre. By Theorem KBD there are $t \in A_N$ and infinite $\mathbb{M}_t$ such that $x_m + t \in A_N$ for $m \in \mathbb{M}_t$. For $m \in \mathbb{M}_t$ one has

$$||f_m(x_m)|| = ||f_m(x_m + t) - f_m(t)|| \leq ||f_m(x_m + t)|| + ||f_m(t)|| \leq 2N.$$

So $\{||f_m(x_m)|| : m \in \mathbb{M}_t\}$ is bounded, a contradiction, since $\mathbb{M}_t$ is infinite (and $||f_m(x_m)|| \geq m\}$ for $m \in \mathbb{M}_t$ ). □

**4. Shift-compactness: a decade (or 67 years) on**

We introduce a more general shift-compactness theorem after some preliminaries. These results are taken from [Ost2013b][2], and with the exception of Theorem 4.3, have brief proofs meriting repetition. They enable the development in Section 6 of the Effros Mapping Principle, an important 'cousin' (non-linear generalization) of the celebrated Open Mapping Theorem (below), giving by far the simplest and shortest proof, as compared to e.g. that given by Ancel [Anc1987].

**Definitions.** 1. As in [BinO2010c, §2], denote by $\mathcal{H}(X)$ the normed group of *bounded* autohomeomorphisms of a space $X$ metrized by $d^X$ comprising those $h(.)$ with finite norm:

$$||h|| := \sup_x d^X(h(x), x) < \infty.$$

2. Say that a subgroup $G \subseteq \mathcal{H}(X)$ *separates* (individual) *points and closed nowhere dense sets* in the space $X$ if, for each $x \in X$ and $F$ closed and nowhere dense in $X$, there is in each neighbourhood of the identity $1_G$ an element $g \in G$ such that $g(x) \notin F$.
3. Say that the above separation of $x$ from $F$ is *strong* if in each neighbourhood of the identity there is a non-empty open set $H$ such that $h(x) \notin F$ for every $h \in H$. Equivalently (when the group is right-topological, as will be the case below), in each open neighbourhood $U$ of $1_G$ there are $g \in U$ and an open neighbourhood $V$ of $1_G$ with $Vg \subseteq U$ and with $Vg(p)$ disjoint from $F$.
4. The group $G \subset \mathcal{H}(X)$ acts *transitively* on a space $X$ if for each $x, y$ in $X$ there is $g$ in $X$ such that $g(x) = y$.

[2] An early survey of shift-compactness was presented at the $25^{\text{th}}$ Summer Topology Conference in July 2010 in Kielce. The current text is thus a review of shift-compactness a decade on.

5. A group $G$ *acts continuously* under $\varphi$ on $X$ if $\varphi : (g, x) \mapsto g(x) := \varphi(g, x)$ is continuous. Here $gh(x) = g(h(x))$. It acts *separately continuously* if each point evaluation $\varphi_x : g \mapsto g(x)$ and each $g : x \mapsto g(x)$ is continuous.

**Lemma 4.1.** *If $G$ is a separable normed group, acting continuously and transitively on a non-meagre space $X$, then for each non-empty open $U$ in $G$ and each $x \in X$ the set $Ux$ is non-meagre in $X$.*

**Proof.** We first work in the right norm topology of $G$. Suppose that $u \in U \subseteq G$ and so without loss of generality assume that $U = B_\varepsilon(u) = B_\varepsilon(1_G)u$ (for some $\varepsilon > 0$); put $y := ux$ and $W = B_\varepsilon(1_G)$. Then $Ux = Wy$. Next work, exceptionally, in the left norm topology (for which $W = B_\varepsilon(1_G)$ is a nhd of $1_G$). As each set $hW$ for $h \in G$ is now open (since now the left shift $g \mapsto hg$ is norm-continuous and so a homeomorphism), the open family $\{gW : g \in G\}$ covers $G$, and so has a countable sub-cover, $\{g_nW : n \in \mathbb{N}\}$ say. As $G$ is transitive, $X = Gy$, and so $X$ is covered by $\{g_nWy : n \in \mathbb{N}\}$. So for some $n$, the set $g_nWy$ is non-meagre. As $g_n^{-1}$ is a homeomorphism of $X$, the set $Wy = Ux$ is also non-meagre in $X$. □

**Remark.** If for each $x$ the point evaluation map $g \mapsto g(x)$ is continuous and $X$ is analytic, then also $Ux$ is analytic and so Baire.

**Lemma 4.2** [Separation Lemma]. *If $G$ is a separable normed group, acting separately continuously and transitively on a non-meagre space $X$, then for any fixed point $x$ and $F$ closed nowhere dense the set*

$$W_{x,F} := \{\alpha : \alpha(x) \notin F\}$$

*is dense open. In particular, $G$ separates points from nowhere dense closed sets.*

**Proof.** The set $W_{x,F}$ is open, as it takes the form $\varphi_x^{-1}(X\backslash F)$ and $\varphi_x(g) = g(x)$ is continuous (by assumption of separate continuity). By Lemma 4.1, for $U$ any non-empty open set in $G$, the set $Ux$ is non-meagre, and so $Ux\backslash F$ is non-empty, as $F$ is meagre. But, then for some $u \in U$ we have $u(x) \notin F$. □

**Theorem 4.3.** *Suppose a separable subgroup $G$ of $\mathcal{H}(X)$ acts transitively on a non-meagre space $X$. Then $G$ strongly separates points and closed nowhere dense sets.*

**Proof.** Here the action is separately continuous, as for each $g \in G$ the autohomeomorphism $g \mapsto g(x)$ is continuous in $x$, and since for fixed $p \in X$,

$$d(g(p), h(p)) \leq \sup_X d(g(x), h(x)) = \sup_X d(g(h^{-1}(x)), x) = ||gh^{-1}||,$$

on substituting $h^{-1}(x)$ for $x$. Now consider a point $p \in X$ and a nowhere dense closed set $F$. By Lemma 4.2 $G$, being a separable normed group acting separately continuously and transitively on a non-meagre space $X$, separates $p$ and $F$. So there is $\tau \in G$ with $\tau(p) \notin F$. Let $\varepsilon := d(\tau(p), F) > 0$. Then for any $\delta < \varepsilon$, we have $B_\delta(1)\tau(p) \subseteq X \backslash F$. Otherwise, $\eta\tau(p) = y \in F$ for some $y$ and $\eta \in B_\delta(1)$, and so

$$\varepsilon = d(\tau(p), F) \leq d(y, \tau(p)) = d(\eta\tau(p), \tau(p)) \leq \sup_X d(\eta(x), x) = ||\eta|| < \delta,$$

a contradiction. That is, $G$ strongly separates. □

The next result originates in [Ost2013b], generalizing its Euclidean predecessor in [MilO2012], and then remerges in [MilMO2021] in a 'constructive' Euclidean presentation (i.e. using effective coding) enabling the use of Gödel's Axiom of Constructibility ($V = L$), to which we return later in this section. Below is a clearer, shorter, and somewhat different proof from that in [Ost2013b]. The end purpose is to enable an easy check on its constructive character in Theorem 4.8 below. The general idea in Theorem 4.4 is that, as more points $u_i$ in $U$ are considered, progressively shorter 'links' are added to a chain of shifts, each ensuring preservation both of membership in an open set $U$ and of separation from a closed set $F$. When the points $u_i$ converge, the existence of a limiting KBD shift comes from Baire's theorem (in Th. 4.6). For a simpler result in a right-topologial group acting on iself, see [MilMO2021, Lemma 1].

**Theorem 4.4** [Finitary Strong Separation]. *Suppose the subgroup $G$ of $\mathcal{H}(X)$ strongly separates points from closed nowhere dense sets of $X$. Let $U \subseteq X$ be open, $u_i \in U$ for $i \leq n$ and $F \subseteq X$ closed and nowhere dense. Then, for each $\varepsilon > 0$, in $B_\varepsilon(1_G)$ there is a non-empty open set $V \subseteq G$ of homeomorphisms $\eta$ such that $\eta(u_i) \in U$ and $\eta(u_i) \notin F$ for each $i \leq n$.*

**Proof**. We use induction on the number of points. The starting step $n = 1$ (which reasserts strong separation) is similar to but simpler than the inductive step, to which we now turn.

Let $\varepsilon > 0$. Take $\bar{\varepsilon} = \min\{\varepsilon/2, d(u_{n+1}, X\backslash U)\}$. By the inductive hypothesis, for the first $n$ point $u_i$, there is a set $V \subseteq B_{\bar{\varepsilon}}(1_G)$ as in the statement of the theorem, i.e. for some $\tau_1 \in V$ with $||\tau_1|| < \bar{\varepsilon}$ and all $\eta \in B_\delta(1_G)$

$$\eta\tau_1(u_i) \in U \text{ and } \eta\tau_1(u_i) \notin F \text{ for } i = 1, ..., n.$$

By the assumed strong separation of $\tau_1(u_{n+1})$ from $F$, choose $\tau_2$ in $B_\delta(1_G)$ and $B_2$ a nhd of $1_G$ with both $B_2\tau_2 \subseteq B_\delta(1_G)$ and $\eta\tau_2\tau_1(u_{n+1}) \in U\backslash F$, for each $\eta \in B_2$.

Take $\tau := \tau_2\tau_1$. Since $||\eta\tau(u_{n+1})|| \leq ||\eta|| + ||\tau|| < \bar{\varepsilon} + \delta < \varepsilon$,

$$\eta\tau(u_{n+1}) \in U.$$

Moreover, for $\eta \in B_2 \subseteq B_\delta(1_G)$,

$$\eta\tau(u_i) \in U \text{ and } \eta\tau(u_i) \notin F \text{ for } i = 1, ..., n.$$

Combining yields the assertion for $n + 1$ points. □

**Remark.** The inductive argument above (key to Theorem 4.6 below) is based on a small 'nudging' of an additional point $u_{n+1}$ away from a nowhere dense set. This can also be developed in a Haar-measure context using the density topology, for which the nowhere dense sets are the null sets, cf. Theorem 8.8 (Kodaira's Theorem). For the details see [MilMO2021, Th. 2H].

**Lemma 4.5**. *Suppose the subgroup $G$ of $\mathcal{H}(X)$ strongly separates points from closed nowhere dense sets of $X$. For $K = \{x_n : n = 0, 1, 2, ..\} \subseteq X$ with $x_n \to x_0$ and closed nowhere dense $F$, the set*

$$W_{K,F} := \{\alpha \in G : \alpha(K) \notin F\} = \{\alpha : \alpha(K) \subseteq X\backslash F\}$$

*is dense open in the norm topology.*

*In particular, for $T \subseteq X$ co-meagre and $G$ a separable subgroup of $\mathcal{H}(X)$ acting transitively on $X$, there is $\alpha \in G$ with*

$$\{\alpha(x_n) : n = 0, 1, 2, ..\} \subseteq T.$$

**Proof.** As to open-ness, for $u \in W_{K,F}$ one has $u(K) \subseteq X\backslash F$, so $\varepsilon := \min_{k \in K}\{d(u(k), F)\} > 0$, as $K$ is compact. Then $Bu \subseteq W_{K,F}$ for $B = B_\varepsilon(1_G)$.

As to density, fix $u$ and write $u_n := u(x_n)$. By Lemma 4.2 we may assume $u(x_0) \notin F$. Now for some $\varepsilon > 0$ and integer $N$ one has $B_\varepsilon(u(x_0)) \subseteq X\backslash F$ and $u(x_n) \in B_{\varepsilon/2}(u(x_0))$ for $n > N$. As in Theorem 4.3 find $\eta$ with $||\eta|| < \varepsilon/2$ such that $\eta(u_i) \notin F$ for $i \leq N$. But for $n > N$ one has $\eta(u(x_n)) \in B_{\varepsilon/2}(u(x_n)) \subseteq B_\varepsilon(u(x_0)) \subseteq X\backslash F$. Thus for all $n$ one has $\eta(u(x_n)) \notin F$, as required.

For the particular case given, Theorem 4.3 applies yielding the required strong separation. Now w.l.o.g. take $T = \bigcap_n (X\backslash F_n)$ with each $F_n$ closed and nowhere dense. Here $G$ is a Baire space (since e.g. the open set $X\backslash F_1$ is non-meagre). So by the above

$$C := \bigcap\nolimits_n \{\alpha \in G : \alpha(K) \subseteq X\backslash F_n\}$$

is a dense $\mathcal{G}_\delta$-set in $G$. Thus

$$\alpha(K) \subseteq T$$

for many (densely-$\mathcal{G}_\delta$ many) $\alpha \in C$. □

For a simpler result in which a Baire right-topological group acts on itself, see [MilMO2021, Th. 1].

We now obtain a first generalization of Theorem KBD above (followed by Theorems 7.7, 8.4 and 8.7). Here shift-compactness is a consequence of separation properties.

**Theorem 4.6.** *For $T$ a Baire non-meagre subset of a metric space $X$ and $G$ a separable normed group, Baire in its right norm topology, acting separately continuously and transitively on $X$:*

*for every convergent sequence $x_n \to x$ there are $\tau \in G$ and an integer $N$ such that $\tau(x) \in T$ and*

$$\{\tau(x_n) : n > N\} \subseteq T.$$

*In particular, this assertion holds in a separable subgroup $G$ of $\mathcal{H}(X)$ acting transitively on a non-meagre space $X$.*

**Proof.** Write $T := M \cup U \backslash \bigcup_n F_n$ with $U$ open, $M$ meagre and each $F_n$ closed and nowhere dense in $X$. Let $u_0 \in T \cap U$. By transitivity there is $\sigma \in G$ with $\sigma x = u_0$. Put $u_n := \sigma x_n$. Then $u_n \to u_0$. Take $K = \{u_n : n = 0, 1, 2, ..\}$. Next write

$$C := \bigcap\nolimits_n \{\alpha : \alpha(K) \notin F_n\},$$

which is a dense $\mathcal{G}_\delta$ by Lemma 4.2. So, as the action is (separately) continuous and $G$ is Baire, the set

$$\{\alpha : \alpha(u_0) \in U\} \cap C$$

is non-empty. For $\alpha$ in this set,

$$\alpha(u_0) \in U \backslash \bigcup\nolimits_n F_n.$$

Now $\alpha(u_n) \to \alpha(u_0)$, by (separate) continuity of $\alpha$, and $U$ is open. So for large enough $n$, $\alpha(u_n) \in U$. Since $\{\alpha(u_m) : m = 1, 2, ..\} \in X \backslash \bigcup_n F_n$, we have that $\alpha(u_n) \in U \backslash \bigcup_n F_n \subseteq T$ for such $n$.
Finally, put $\tau := \alpha\sigma$. Then $\tau(x) = \alpha\sigma(x) \in T$ and $\{\tau(x_n) : n > N\} \subseteq T$.

For the final assertion, by Theorem 4.3 $G$ is a separable Baire group acting separately continuously and transitively on $X$, as in Lemma 4.5. □

A refinement of this argument (for which see [Ost2015]) involves a second Baire non-meagre set providing location for the 'shifter' $\tau$ in the preceding result:

**Theorem 4.7** [Baire Shift Theorem]. *For $T$ a Baire non-meagre subset of a metric space $X$ and $G$ a separable normed group, Baire in its right norm topology, acting separately continuously and transitively on $X$:*

*for every convergent sequence $x_n$ with limit $x$ and any Baire non-meagre $A \subseteq G$ with $1_G \in A$ such that $Ax \subseteq T$, there are $\alpha \in A$ and an integer $N$ such that $\alpha x \in T$ and*

$$\{\alpha(x_n) : n > N\} \subseteq T.$$

*Constructive versions.*

We recall the Gödel constructible hierarchy $\langle L_\alpha : \alpha \in On\rangle$, where $On$ denotes the class of all ordinals and, for each ordinal $\alpha$, the sets $L_\alpha$ are obtained by iterating transfinitely the operation which defines $L_{\beta+1}$ as the family of subsets of $L_\beta$ definable by the first-order formulas in the formal language of set theory (i.e. with primitive symbol $\in$ for membership). Here the sets are allowed formal definitions that refer to a finite string of elements of $L_\beta$ and all of their quantifiers range over $L_\beta$ – see eg. [BinO2019a]. The class $L := \{L_\alpha : \alpha \in On\}$ comprising all the constructible sets has a canonical well-ordering $<_L$ (defined by transfinite induction using an effective listing of all of the countably many predicates of the formal language).

Given a separable group $X$ with a constructible listing of a dense set $D$ and corresponding listing $I_1, I_2, ...$of the basis of open balls $\{B_{1/n}(d) : d \in D\}$, one considers the (cumulative) class $\mathcal{G}_\alpha$ of open sets coded by $L_\alpha$, i.e.

$$\mathcal{G}_\alpha = \{U : U = \bigcup\nolimits_n I_{x_n} \text{ for some sequence } x_n \text{ in } L_\alpha\}.$$

The family of complements of sets in $\mathcal{G}_\alpha$ is denoted by $\mathcal{F}_\alpha$.

We now have the tools to define *constructive strong separation* by requiring in the definition of strong separation as above that the set $H$ is in the family $\mathcal{G}_\alpha$ for some ordinal $\alpha$. Note that we do not presume members of $X$ and so of $H$ are in $L_\alpha$, only that the open set is coded by $L_\alpha$. Recall that by the definition of group action

$$(gh)(p) = g(h(p)).$$

Let $D$ dense in $G$ have a constructible enumeration. Given $p$ (with $p \in L_\alpha$) and $F$ closed nwd also coded in $L_\alpha$ there is a least $n \in \mathbb{N}$ with

$$B_{1/n}(1_G)p = \{g(p) : ||g|| < 1/n\} \text{ disjoint from } F.$$

Using its constructible enumeration, pick the 'first' $d \in D \cap B_{1/n}(1_G)$. Then, as in the proof above, for some least $m \in \mathbb{N}$

$$B_{1/m}(d) = B_{1/m}(1_G)d \subseteq B_{1/n}(1_G).$$

So

$$B_{1/m}(1_G)dp = \{(gd)(p) : ||g|| < 1/m\} \text{ is disjoint from } F.$$

From here it follows that Theorem 4.3 has the constructible version below. For this, recall that a canonical non-meagre Baire set is a non-meagre $\mathcal{G}_\delta$-set. (Remove any surplus meagre set by expansion to a countable union of closed nowhere dense sets.)

**Theorem 4.8** [Constructible KBD; cf. [MilMO2021, Th. 1E]]. *Suppose the subgroup $G$ of $\mathcal{H}(X)$ is separable and acts transitively on the non-meagre metric space $X$, and that both $G$ and $X$ are coded in $L_\alpha$. Given any convergent sequence $x_n \in L_\alpha$, and non-meagre $\mathcal{G}_\delta$-set $T = \bigcap G_n \subseteq X$ also coded in $L_\alpha$, there is $h \in G$ coded in $L_{\alpha+\omega}$ and $N \in \mathbb{N}$ with*

$$\{h(x_n) : n > N\} \subseteq T.$$

Theorem1E of [MilMO2021] is a corollary, since the group of Euclidean translations is normed exactly as Euclidean space: for $h(x) = c + x$,

$$||h|| = ||(c + x) - x|| = ||c||.$$

(The fact that this group strongly separates points from closed nowhere dense sets was shown ibid. Prop. 1.) Constructible versions are relevant to the construction of singular sets from Gödel's Axiom of Constructibility, $V = L$.

**5. Subgroup theorems**

There are two well-known dichotomies (here 'small or large', rather than 'nice or nasty', as in the Darboux dichotomy where additive functions are either continuous or everywhere unbounded, cf. [BinO2011]) asserting that a Baire subset is *either meagre or clopen.* From our current perspective they are 'duals' (as with the generalization of Theorem KBD). There is the *Banach-Kuratowski dichotomy,* that a Baire subgroup is either meagre or clopen ([Ban1931, Satz 1],[Kur1966, Ch VI, 13 XII], cf. [Kel1955, Ch 6 Prob P], [BGT Cor 1.1.4], [BecCS1958], [Bec1960]), where the context is a group $G$ and the subset is a subgroup $H$ (evidently invariant under the translation action of $H$). There is also the *Kuratowski-McShane dichotomy* [Kur1966], [McS1950, Cor.1], concerning now a Baire subset $S$ of a topological space being either meagre or clopen, with the premise of a transitive action of a family of autohomeomorphisms such that each action either leaves the subset $S$ invariant or shifts it into disjointness (see [BinO2010c]).

The dichotomies below are in keeping with this, though they interpret large as 'total'. We give an application in Theorem 5.3 below – see [BinO2011] for others related to additivity, sub-additivity, and convexity (or for more detailed analyses: [BinO2008], [BinO2009], [BinO2010a]). As may be expected from the Banach-Kuratowski dichotomy, for totality one relies either on density or connectedness. The following direct proofs are inspired by a close reading of work by Hoffmann-Jørgensen ([Hof1980, p. 255]), where the subgroup theorem is implictly used for a topological group. They assume less than completeness.

**Theorem 5.1** [Subgroup Theorem - density version]. *In a complete normed group $G$, if $H$ is a dense non-meagre subgroup with the Baire property, then $H = G$.*

**Proof.** We interpret the statement in the right norm topology. By the Steinhaus-Weil theorem, Theorem 9.1 below, since $H$ is Baire non-meagre and a subgroup, $H^{-1}H$ is an open nhd of $1_G$, and so $H = H^{-1}H$ is also an open nhd of $1_G$ in $G$. For any $g \in G\backslash H$ one has $H \cap Hg = \emptyset$ (as otherwise $h_1 = h_2 g$ for $h_1, h_2 \in H$ implies $g = h_2^{-1}h_1 \in H$), and so $Hg$ is a nhd of $g$ avoiding $H$. So $H$ is closed in $G$; so, being dense in $G$, it is the whole of $G$.$\square$

Our own earlier normed-group approach in [BinO2010c] relied on a weakening of the *Archimedean* property (i.e. $G = \bigcup_{n\in\mathbb{N}} H^n$) in lieu of density to derive a similar result, whereas in [BinO2011] we used Kronecker's Theorem to show that in the additive group $\mathbb{R}$ a non-meagre subgroup is dense. In the absence of density the argument above still goes through when the group is connected, as then the Archimedean property holds for $H$, as we show below.

**Theorem 5.2** [Subgroup Theorem – connected group version]. *In a connected complete normed group $G$, if $H$ is a non-meagre subgroup with the Baire property, then $G = \bigcup_{n\in\mathbb{N}} H^n$ and so $H = G$.*

**Proof.** As in the proof above, $H^{-1}H = H$ is an open nhd of $1_G$ in $G$. Suppose that $B := B_\varepsilon(1_G) \subseteq H$. As $B$ is symmetric and $BS = \bigcup_{s\in S} Bs$ is open for any set $S$, the set $C := \bigcup_{n\in\mathbb{N}} B^n$ is an open subgroup with $C \subseteq H$. Now for any $g \in G\backslash C$ one has $C \cap Cg = \emptyset$ (as before, since otherwise $c_1 = c_2 g$ for $c_1, c_2 \in C$ implies $g = c_2^{-1}c_1 \in C$), and so $Cg$ is a nhd of $g$ avoiding $C$. So the non-empty set $C$ is both closed and open in $G$, and, by the connectedness of $G$, is the whole of $G$, i.e. $G = C = H$. $\square$

**Theorem 5.3** [Loy [Loy1976], Hoffmann-Jørgensen [Hof1980]]. *A non-meagre analytic topological group is Polish.*

**Proof.** An analytic topological group $H$, being separable, may be densely embedded (by completion) in a complete separable topological group $G$, but now $H$ is a non-meagre subgroup with the Baire property (being analytic), so is all of $G$ by Theorem 5.1. $\square$

We next recall the classical Open Mapping Theorem from linear functional analysis, including its analytic sets proof, as it is so short, (cf. [Rud1991, 48-49], [Con1990, 12.1]) and as a prelude to its non-linear generalization Theorem 6.1.

**Theorem 5.4 (Open Mapping Theorem)**. *If $L : E \to F$ is a linear, continuous surjection between Fréchet spaces, then $L$ is an open mapping (takes open sets to open sets).*

**Proof.** [Ost2015] Consider $L : E \to F$, a linear, continuous surjection between Fréchet spaces, and $U$ a nhd of the origin. Choose a small enough open nhd $A$ of the origin with $A - A \subseteq U$. As $L(A)$ is non-meagre (since $\{nL(A) : n \in \mathbb{N}\}$ covers $F$) and, being the continuous image of an open (so analytic) set is Baire, $L(A)-L(A)$ is a nhd of the origin by the Steinhaus-Weil Theorem (Th. 9.1 below). But of course

$$L(U) \supseteq L(A) - L(A),$$

so $L(U)$ is a nhd of the origin. So $L$ is an open mapping. □

**6. Effros Open Mapping Principle**

We again follow the development in [Ost2013b]; cf. [Ost2013a], [Ost2015]. Recall that $\mathcal{H}(X)$ is the group of *bounded* self-homeomorphisms of a metric space $X$ onto itself, comprising those $h(.)$ such that $\sup_x d^X(h(x), x) < \infty$. A subgroup $G$ acts *microtransitively* on $X$ if for $U$ open in $G$ and $x \in X$ the set $\{h(x) : h \in U\}$ is a neighbourhood of $x$; we refer to §4 for other terms concerning group action.

Theorem E below is a non-linear extension of the familiar Open Mapping Theorem of functional analysis (see e.g. Rudin [Rud1991, §2.10-12]), recalled in Theorem 5.4 above.

**Theorem 6.1 (Theorem E – Effros Open Mapping Principle).** *Let the normed group $G$ have separately continuous and transitive action on $X$. If under either norm topology $G$ is analytic and Baire and $X$ is non-meagre, then the action of $G$ is microtransitive. That is, for $U$ an open neighbourhood of $1_G$ and for arbitrary $x \in X$, the set $Ux := \{u(x) : u \in U\}$ is a neighbourhood of $x$. So the point-evaluation maps $g \mapsto g(x)$ are open for each $x$.*

**Remark.** The proof below again refers to analyticity multiple times in order to use the fact that the continuous image of an analytic set is analytic and so has the Baire property (again Nikodym's Theorem, cf. §3). The short proof follows that in [Ost2013b] but one hopes made clearer with more explicit justification at key steps.

**Proof of Theorem 6.1.** Assume $G$ acts transitively on $X$ and that $X$ is non-meagre. Let $B := B_\varepsilon(1_G)$ and suppose that, for some $\varepsilon > 0$ and some $x$, the set $T := Bx$ is not a nhd of $x$. Then there is $x_n \to x$ with $x_n \notin Bx$ for each

$n$. Take $A := B_{\varepsilon/2}(1_G)$ so that $1_G \in A$, and note that $A$ is a symmetric open set ($A^{-1} = A$, by the inversion axiom (ii) for norms in §2). By Lemma 4.1, $Ax$ is non-meagre, since $G$ is separable. ($G$ being analytic, is Lindelöf [RogJ1980, §2.7], so being metrizable is separable.) Being open, $A$ is analytic and, since by separate continuity the evaluation map $g \mapsto g(x)$ is continuous, also $Ax$ is analytic and so again Baire. By Theorem 4.6 above, as $Ax \subseteq Bx = T$, there are $a \in A$ (which, being open is Baire non-meagre, allowing $A$ as a location for $a$) and a co-finite $\mathbb{M}_a$ such that $ax_m \in Ax$ for $m \in \mathbb{M}_a$. For any such $m$ choose $b_m \in A$ with $ax_m = b_m x$. Then $x_m = a^{-1}b_m x \in A^2 x \subseteq Bx$, a contradiction (note that $a^{-1} \in A$, by symmetry). □

**7. 'Bunching': intersecting shifts**

In this section we identify an implicit use by Parthasarathy of our variant of shift-compactness and then show in Theorem 7.7 a connection with a very general shift-compactness theorem. Below $K$ always refers to a compact set in a fixed metric group $G$ wherein $B_\delta$ denotes the open ball of radius $\delta > 0$, centred at $1_G$.

If a (compact) set $K$ meets a right-shift of itself, $Kx$ say, then $x \in K^{-1}K$. In particular, if this occurs for all shifts $x$ in some set, then one may regard the set as witnessing a 'bunching' phenomenon. For instance, if the witnessing set comprises all sufficiently small shifts $x$, in whatever sense, e.g. those in the open ball $B_\delta$, then $1_G \in B \subseteq K^{-1}K$. This makes $1_G$ be an interior point of $K^{-1}K$, exactly as in the Steinhaus-Weil Interior Point Theorem, Theorem 9.1 below. In view of the many applications of the latter theorem, it is of interest to identify appropriate notions of smallness and circumstances guaranteeing bunching. The ball $B$ can in certain circumstances be relatively open, for instance in the *Cameron-Martin subspace* of a Hilbert space [BinO2019b], or as in the Borell interior point theorem [Bor1976].

An example of this occurs in the following result.

**Theorem 7.1** ([Par1967, Ch. 3 Th. 3.1]; [Hey1977, Th. 1.2.10 p.34]). *For $X$ a Polish group, if $\mu$ is an idempotent probability measure on $X$, then its support is a compact subgroup on which $\mu$ is normalized Haar measure.*

It is of interest here that Weil's proof of SW [Wei1940] uses the convolution of the indicator functions of $K$ and $K^{-1}$, again a bunching phenomenon; see [HewR1979, p. 296]. The bunching in the case of the idempotent measure above is a consequence of Lemma 7.2. We omit the proof.

**Lemma 7.2** ([Par1967, Ch. 3 Lem. 3.1]). *For $X$ a Polish group and $K$ a compact subset, if $\mu$ a probability measure on $X$, then for some $x_0 \in X$*

$$\mu(Kx_0) = \delta_K := \sup_{x \in X} \mu(Kx).$$

Indeed, for $\delta_K > 0$, and any sequence $x_n$ with $\mu(Kx_n) \to \delta_K$ with $\mu(Kx_n) \geq \delta_K/2$, there cannot be a subsequence $\mathbb{M} \subseteq \mathbb{N}$ with $\{Kx_m : m \in \mathbb{M}\}$ mutually disjoint. In fact, as noted earlier in §3, if one may choose $\mathbb{M} \subseteq \mathbb{N}$ with $\bigcap_{m \in \mathbb{M}} Kx_m$ non-empty, then for $t$ in this intersection one has

$$\{tx_m^{-1} : m \in \mathbb{M}\} \subseteq K,$$

reproducing our variant of shift-compactness. Indeed, very similar ideas can be traced back to the Simmons-Mospan converse of SW, for which see §9.

It emerges that a related measure subcontinuity effect (below) produces similar bunching, highly relevant to the Steinhaus-Weil Theorem, and links *amenability at 1* with *shift-compactness,* for which see Theorem 7.7 below.

**Definition** ([BinO2020a]). For $\mu$ a probability measure (or, more generally, a finitely additive measure) and compact $K$, noting that $\mu_\delta(K) := \inf\{\mu(Kt) : t \in B_\delta\}$ is weakly decreasing in $\delta$, put

$$\mu_-(K) := \sup_{\delta > 0} \inf\{\mu(Kt) : t \in B_\delta\},$$

and, for $\mathbf{t} = \{t_n\}$ a *null sequence*, i.e. with $t_n \to 1_G$,

$$\mu_-^{\mathbf{t}}(K) := \liminf\nolimits_{n \to \infty} \mu(Kt_n).$$

Then

$$0 \leq \mu_-(K) \leq \mu(K) = \inf_{\delta > 0} \sup\{\mu(Kt) : t \in B_\delta\}.$$

(See [[BinO2020a, Prop. 1].) We say that a null sequence $\mathbf{t}$ is *non-trivial* if $t_n \neq 1_G$ infinitely often. Define as follows:
(i) $\mu$ is *translation-continuous* (‘*continuous*’ or ‘*mobile*’) if $\mu(K) = \mu_-(K)$ for all compact $K \subseteq G$;
(ii) $\mu$ is *subcontinuous* if $0 < \mu_-(K) \leq \mu(K)$ for all non-null compact $K \subseteq G$;
(iii) $\mu$ is *(selectively) subcontinuous at a non-null compact $K$ along* $\mathbf{t}$ if $\mu_-^{\mathbf{t}}(K) > 0$.

For compact $K$ and $\delta, \Delta > 0$, put

$$B_\delta^\Delta = B_\delta^{K,\Delta}(\mu) := \{z \in B_\delta : \mu(Kz) > \Delta\},$$

which is monotonic in $\Delta : B_\delta^\Delta \subseteq B_\delta^{\Delta'}$ for $0 < \Delta' \leq \Delta$. Note that $1_G \in B_\delta^\Delta$ for $0 < \Delta < \mu(K)$. It is precisely these sets, when aggregated by reference to a a suitable selectively subcontinuous measure, that create interior points 'by bunching together' as in Lemma 7.3 below. The specialization below to a mobile measure (see above) may be found in [Gow1970], [Gow1972]. Here and below we denote by $\mathcal{M}_+(\mu)$ the $\mu$-non-null measurable sets.

**Lemma 7.3** (cf. [BinO2020a])**.** *Let $\mu$ be a finitely additive measure on a metric group $G$. For $\mu$-non-null $K$, if $\mu_-^{\mathbf{t}}(K) > 0$ for some non-trivial null sequence $\mathbf{t}$, then for $\Delta \geq \mu_-^{\mathbf{t}}(K)/4 > 0$ there is $\delta > 0$ with $t_n \in B_\delta^\Delta$ for all large enough $n$ and*

$$\Delta \leq \mu(K \cap Kt) \qquad (\text{for all } t \in B_\delta^\Delta),$$

*so that*

$$K \cap Kt \in \mathcal{M}_+(\mu) \qquad (\text{for all } t \in B_\delta^\Delta). \tag{$*$}$$

*In particular,*

$$K \cap Kt \neq \emptyset \qquad (t \in B_\delta^\Delta),$$

*or, equivalently,*

$$B_\delta^\Delta \subseteq K^{-1}K, \tag{$**$}$$

*so that $B_\delta^\Delta$ has compact closure.*

*A fortiori, if $\mu_-(K) > 0$, then $\delta, \Delta > 0$ may be chosen with $\Delta < \mu_-(K)$ and $B_\delta \subseteq B_\delta^\Delta$ so that (*) and (**) hold with $B_\delta$ replacing $B_\delta^\Delta$, and in particular $G$ is locally compact.*

Lemma 7.3 applies in particular to invariant measures (as there $\mu_- = \mu$), so that a group $G$ with a Haar measure is necessarily locally compact, and conversely by Haar's theorem. Thus $G$ has a Haar measure iff $G$ is locally compact. However, one can relax local compactness, at the expense of introducing pathology in the measure. We refer to [DieS2014, Ch.10] for detail here.

Recall that a group $G$ is *amenable at* 1 [Sol2006] (see below for the origin of this term) if, given a sequence of probability measures $\boldsymbol{\mu} := \{\mu_n\}_{n\in\mathbb{N}}$ with

$1_G \in \mathrm{supp}(\mu_n)$, for $n \in \mathbb{N}$ there are probabilities $\sigma$ and $\sigma_n$ on $G$ with $\sigma_n \ll \mu_n$ satisfying

$$\sigma_n * \sigma(K) \to \sigma(K) \qquad (K \in \mathcal{K}(G)).$$

The term 'amenability at 1' (see [Sol2006, end of §2]) should be viewed as a localization, via the restriction that supports contain $1_G$, of a *Reiter-like condition* [Pat1988, Prop. 0.4] which characterizes amenability: for any probability measure $\mu$ and $\varepsilon > 0$, there is a probability measure $\nu$ with

$$|\nu * \mu(K) - \nu(K)| < \varepsilon \qquad \text{(for all compact } K).$$

Abelian Polish groups are amenable at 1: see again [Sol2006].

**Theorem 7.4** (**Subcontinuity Theorem**, after Solecki [Sol2006, Th. 1(ii)]). *For $G$ Polish and amenable at $1_G$ and $\mathbf{t}$ a null sequence, there is a probability measure $\sigma = \sigma(\mathbf{t})$ such that for each compact $K \in \mathcal{M}_+(\sigma)$ there is a subsequence $\mathbf{s} = \mathbf{s}(K) := \{t_{m(n)}\}$ with*

$$\lim_n \sigma(Kt_{m(n)}) = \sigma(K) \qquad (n \in \mathbb{N}), \text{ so} \qquad \sigma_-^{\mathbf{s}}(K) > 0.$$

Now we recall that $E$ is *left Haar null*, abbreviated to $E \in \mathcal{HN}$, as in Solecki [Sol2005], [Sol2006], [Sol2007] (following [Chr1972], [Chr1974]) if there are a universally measurable $B$ covering $E$ and a probability measure $\mu$ with

$$\mu(gB) = 0 \qquad \text{(for all } g \in G).$$

Theorem 7.5 below aggregates all the relevant sets $B_\delta^\Delta$ of the preceding Lemma 7.3, by including all shifts $gK$ of all compact sets $K$ and all null sequences $\mathbf{t}$ using the associated measures $\sigma(\mathbf{t})$ of Theorem 7.4. The notation $\delta(gK, \Delta)$ also refers to the corresponding $\delta$ of Lemma 7.3.

**Theorem 7.5** (**Aggregation Theorem**). *For $G$ Polish and amenable at $1_G$, and $E$ universally measurable but not left Haar null, put*

$$\hat{E} := \bigcup\nolimits_{\Delta>0,g,\mathbf{t}} \{B^{gK,\Delta}_{\delta(gK,\Delta)}(\sigma(\mathbf{t})) : K \subseteq E, 0 < \sigma(\mathbf{t})(gK)/4 \le \Delta < \sigma(\mathbf{t})(gK)\}.$$

*Then*

$$1_G \in \mathrm{int}(\hat{E}) \subseteq \hat{E} \subseteq E^{-1}E.$$

*In particular, for $E$ open, $1_G \in \mathrm{int}(\hat{E})$.*

Key to the proof is a sequence $z_n \in \hat{E}\backslash B_{1/n}(1_G)$ (cf. §3). The immediate consequence is Solecki's theorem of Steinhaus-Weil type:

**Corollary 7.6** (**Solecki's Interior-Point Theorem** [Sol2006, Th 1(ii)]). *For $G$ Polish and amenable at $1_G$, if $E$ is universally measurable and not left Haar null, then $1_G \in \mathrm{int}(E^{-1}E)$.*

Theorem 7.4 and Lemma 7.3 may be used to prove the following shift-compactness result.

**Theorem 7.7** (**Shift-compactness Theorem** for $\mathcal{HN}$). *For $G$ Polish and amenable at $1_G$, if $E$ is universally measurable and not left Haar null and $z_n$ is null, then there are $s \in E$ and an infinite $\mathbb{M} \subseteq \mathbb{N}$ with*

$$\{sz_m : m \in \mathbb{M}\} \subseteq E.$$

*Indeed, this holds for quasi all $s \in E$, i.e. off a left Haar null set.*

The abelian case of Th 7.7 was independently established by Banakh and Jabłońska in [BanaJ2019], where sets that are not shift-compact are termed null-finite: for a wider discussion see [BinO2024, §§15.7,15.8].

**8. Beyond and around groups**

At the heart of shift-compactness is the embedding $(t, z_n) \mapsto tz_n$. But that may be in two ways: as an action of some 'translators' $t$ on a given null sequence, as above, or as an action of a given null sequence $z_n$ on a set of translators. Here in this section we consider the latter view on shift-compactness. We need the following definition.

**Definition** [Category convergence, [BinO2010d]] A sequence of homeomorphisms $\psi_n$ satisfies the *category convergence condition* (cc) if:

For any non-empty open set $U$, there is a non-empty open set $V \subseteq U$ such that, for each $k \in \mathbb{N}$

$$\bigcap\nolimits_{n \geq k} V \backslash \psi_n^{-1}(V) \text{ is meagre.}$$

Equivalently, for each $k \in \mathbb{N}$, there is a meagre set $M$ such that, for $t \notin M$

$$t \in V \Longrightarrow (\exists n \geq k)\psi_n(t) \in V.$$

For this ‘subsequence convergence to the identity’ form, see [BinO2010d]. See [BinO2010d] also for the proofs of Theorems 8.1-8.8 below, unless otherwise specified.

**Theorem 8.1** [Category Embedding Theorem, CET]. *Let $X$ be a topological space. Suppose given homeomorphisms $h_n : X \to X$ for which the category convergence condition (cc) is met. Then, for any non-meagre Baire set $T$ for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

**Theorem 8.2** [First Verification Theorem for category convergence]. *For $(X, d)$ a metric space, if $\psi_n \in \mathcal{H}(x)$ converges to the identity under the norm $||h|| := \sup_x d^X(h(x), x)$, then $\psi_n$ satisfies the category convergence condition (cc).*

To deduce from CET a version of the KBD Theorem we need to note the necessary context:

**Theorem 8.3** [BinO2010c, Th. 6.3]. *Let $X$ be a normed group. Under the right norm topology of the metric $d_R$ the homeomorphisms $\rho_n(x) := xz_n$ converge under $d_R$ to the identity for all $z_n \to 1_G$ iff $X$ is a topological group (i.e. both $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous).*

This leads to another generalization of Theorem KBD.

**Theorem 8.4** [Topological Kestelman-Borwein-Ditor Theorem]. *In a normed topological group $X$, let $z_n \to 1_G$ be a null sequence. If $T$ is a Baire subset of $X$, then for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{z_m t : m \in \mathbb{M}_t\} \subseteq T.$$

*Likewise, for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

**Theorem 8.5** [Piccard-Pettis Theorem – Piccard [Pic1939], [Pic1942], Pettis [Pet1950]]. *In a normed topological group whose norm topology is Baire: for $A$ Baire and non-meagre (in the norm topology), the sets $AA^{-1}$ and $A^{-1}A$ both have non-empty interior.*

As with the real line one may consider the analogue of the Lebesgue density topology, namely the *Haar-density topology* in a locally compact topological group. For the details (which combine results of N. F. G. Martin in 1964 and B. J. Mueller in 1965) see [BinO2024] and [BinO2010c].

**Theorem 8.6** [Second Verification Theorem for category convergence]. *Let $X$ be a locally compact topological group with left-invariant Haar measure $\eta$. Let $V$ be $\eta$-measurable and non-null. For any null sequence $z_n \to 1_G$ and each $k \in N$,*

$$H_k = \bigcap\nolimits_{n \geq k} V \backslash V \cdot z_n$$

*is of $\eta$-measure zero, so is meagre in the Haar density topology $\mathcal{D}$.*

*That is, the sequence $h_n(x) := xz_n^{-1}$ satisfies the category convergence condition (cc) under $\mathcal{D}$.*

From here a measure-theoretic case of Theorem KBD follows:

**Theorem 8.7** [Generalized Kestelman-Borwein Ditor Theorem – Measurable Case]. *Let $X$ a normed locally compact topological group, $z_n \to 1_G$ be a null sequence in $X$. If $T$ is Haar measurable and non-null, then for almost all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{tz_m : m \in \mathbb{M}_t\} \subseteq T.$$

This theorem in turn yields two important conclusions. The first is Kodaira's Th 8.8 below and the second is Th. 9.1 in the next section.

**Theorem 8.8** [Kodaira's Theorem – [Kod1941, Corollary to Satz 18. p. 98], cf. [Com1984, Th. 4.17 p.1182]]. *Let $X$ be a normed locally compact group and $f : X \to Y$ a homorphism into a separable normed group $Y$. Then $f$ is Haar-measurable iff $f$ is Baire under the Haar density topology $\mathcal{D}$ iff $f$ is continuous under the norm topology.*

The situation here is analogous to what is well-known in the context of $\mathbb{R}$ in aligning null measurable sets with the meagre sets (indeed the nwd dense sets) of the Lebesgue density topology: see [Kec1995, 17.47], justifying a bitopological viewpoint and the primacy of category over measure in its group-theoretic setting. Compare the title inversion between Oxtoby [Oxt1980] and [BinO2024].

**9. The Steinhaus-Weil Theorem and its converse**

We begin with the celebrated Steinhaus-Weil Theorem [Ste1920], [Wei1940]; cf. Comfort [Com1984, Th. 4.6 p. 1175], Beck et al. [BecCS1958], [BinO2020a], [BinO2020b], [BinO2021], [BinO2022], [BinO2024]:

**Theorem 9.1** [Steinhaus-Weil Theorem]. *In a normed locally compact group $G$, for $S$ of positive Haar measure the difference sets $SS^{-1}$ and $S^{-1}S$ have $1_G$ as interior point.*

We close with two theorems, a corollary and some remarks concerning the Steinhaus-Weil property of Theorem 9.1 above. This has a converse due to Simmons [Sim1975] and later independently to Mospan [Mos2005]. The latter revisits bunching. See also [BinO2018].

**Theorem 9.2** [The Simmons-Mospan Converse]. *In a locally compact Polish group, a Borel measure has the Steinhaus-Weil property if and only if it is absolutely continuous with respect to Haar measure.*

**Theorem 9.3** [Mospan property]. *For $G$ a metric group and compact $K \in \mathcal{K}(G)$ :*
(i) *if $1_G \notin \mathrm{int}(K^{-1}K)$, then $\mu_-(K) = 0$; equivalently, there is a null sequence $t_n \to 1_G$ with $\lim_n \mu(Kt_n) = 0$;*
(ii) *conversely, if $\mu(K) > \mu_-(K) = 0$, then there is a null sequence $t_n \to 1_G$ with $\lim_n \mu(Kt_n) = 0$, and there is a compact $C \subseteq K$ with $\mu(K \backslash C) = 0$ with $1_G \notin \mathrm{int}(C^{-1}C)$.*

**Corollary 9.4.** *A (regular) Borel measure $\mu$ on a locally compact metric topological group $G$ has the Steinhaus-Weil property iff either* (i) *or* (ii) *holds.*
(i) *for each $K \in \mathcal{K}_+(\mu)$, the map $m_K : t \to \mu(Kt)$ is subcontinuous at $1_G$;*
(ii) *for each $K \in \mathcal{K}_+(\mu)$, there is no 'null' sequence $t_n \to 1_G$ with $\mu(Kt_n) \to 0$.*

The more general Steinhaus-Weil interior points property for the composite case of sets $AB$ with $A, B$ non-null are studied in [BinO2022] and in

Section 14.5 of [BinO2024] (cf. [Kha2019]). In the locally compact (Haar) case, as above, $AB$ will have interior points. In general, however, the simple property does not imply the composite; thus, for instance, Matoŭsková and Zelený [MatZ2003] show that in any non-locally compact abelian Polish group there are closed non-(left) Haar null sets (Section 7) $A, B$ such that $A + B$ has empty interior.

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